Action Models in Inquisitive Logic*

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Abstract

Information exchange can be viewed as a process of asking questions and answering them. While dynamic epistemic logic traditionally focuses on statements, recent developments have been concerned with ways of incorporating questions. One approach, based on the framework of inquisitive semantics, is Inquisitive Dynamic Epistemic Logic (IDEL). In this system, agents are represented with *issues* as well as information. On the dynamic level, it can model actions that raise new issues. Compared to other approaches, a limitation of IDEL is that it can only encode public announcements.

We show how IDEL can be refined to encode private questions, by merging its static basis, Inquisitive Epistemic Logic (IEL), with Action Model Logic (AML). This can be done in two ways, namely by enriching action models with *questions* as possible actions or with *issues* concerning which action takes place. We introduce the corresponding dynamic logics, which are conservative extensions of both AML and IEL, and we give a sound and complete axiomatization for both.

1 Introduction

Dynamic epistemic logics are used to analyse the process of information exchange: they describe situations in which agents learn facts about the world and about each other's knowledge. We often interpret these systems as describing communication or scientific inquiry. However, typical information exchange is not just about making statements. Rather, it is a process of *asking questions* and *answering* them. In a similar way, scientific inquiry is not just about making observations, but also about which questions are asked.

This insight has lead to the development of dynamic epistemic logics with questions ([2, 18, 16, 22, 8][1]), with an eye to applications in epistemology, philosophy of science and linguistics (see the introduction to [13] for an overview).

Two prominent approaches are the Dynamic Logic of Questions (DELQ, [22]) and Inquisitive Dynamic Epistemic Logic (IDEL, [8], [5]). These systems are generalizations of Public Announcement Logic (PAL, [19]): they are used to reason about epistemic events of which all agents are fully aware, but, in contrast to PAL, they allow the contents of these events to be questions as well as statements. In both systems, statements convey information, while questions raise *issues*. But there are also important differences between DELQ and IDEL:

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[1] A related line of work focuses specifically on questions embedded under 'know'. See [25] for an overview.

1. *The formal notion of issues.* Issues in DELQ are modelled as an equivalence relation on the set of worlds, following [10]. This relation induces a partition, of which each cell represents an answer to the issue. IDEL adopts a more general notion of issues from inquisitive semantics ([6], [7]), in which alternatives need not be mutually exclusive and can overlap. As a consequence, IDEL can encode conditional questions like 'If John comes to the party, will Mary come?' and alternative questions like 'Does John speak English, German or French?', while these are beyond the scope of DELQ.

2. *The way in which questions are expressed.* In DELQ, all formulas are statements. The asking of a question is encoded as a special type of action. In IDEL, formulas can be questions as well as statements, and there is only one type of action. The latter setting allows for question-embedding sentences such as 'John wonders whether Mary will come to the party'.

These differences are reasons to favour IDEL over DELQ.[2] However, this comparison is not the full story, since both IDEL and DELQ are limited to public actions. In standard DEL, the generalization from public actions to general actions, of which not every agent is equally aware, is made by describing actions using *action models*, resulting in Action Model Logic (AML, [3]).

When we enrich AML with questions, we can reason about scenarios in which participants are not fully aware of what is being stated or asked. When this happens, they become uncertain about each other's knowledge and issues. DELQ has a non-public variant ELQm to encode such situations.[3] As of yet, such a variant is lacking for IDEL.

In this paper, we show how IDEL can be refined to also encode private epistemic actions, by merging its static basis, Inquisitive Epistemic Logic (IEL, [8]), with AML. Table 1 shows in a schematic way how the logics discussed so far are related; the goal of this paper is to fill the empty cell.

	No questions	Questions based on issue logic	Questions based on inquisitive semantics
Static	EL	ELQ	IEL
Dynamic with public announcements	PAL	DELQ	IDEL
Dynamic with public and private announcements	AML	ELQm	

Table 1: Overview of standard epistemic logics, issue logics and inquisitive epistemic logics

We will proceed as follows. We start by introducing IEL and some relevant notions in Section 2. We will see that there are two natural approaches to our goal. The first approach is worked out in detail in Section 3 and we sketch the second one in Section 4. We compare and discuss the approaches in Section 5 and conclude in Section 6.

[2] For an extensive discussion, see [8].

[3] Although, as we will see in Section 5.3, the language of ELQm does not provide an adequate way to express that an agent is uncertain about another agent's issues.

2 Background

2.1 Inquisitive Epistemic Logic

2.1.1 Introducing IEL

In inquisitive logic, questions and statements are treated in a uniform way, by shifting the evaluation of formulas from single worlds to *sets of worlds*. Sets of worlds are *information states* and the meaning of formulas is viewed in terms of the information states that support them, rather than the worlds in which they are true.

For instance, the question 'Is John home?' is supported by the information states that specify whether John is home. All of these states specify either that John is home or that he is not. Thus, there are two maximal supporting states (states that are not contained by some other state), namely the state consisting of all the worlds in which he is home and the one consisting of all the worlds in which he is not. In this way, the meaning of the sentence presents alternative ways of supporting it. Because of this, it is called a *question*.

In inquisitive epistemic models, an agent does not just have an information state in each world, but also an *inquisitive state*. This is the set of information states that count as resolving the agent's issues. It is always downward closed, which means it is closed under subsets. This requirement makes sure that if some body of information is enough to resolve an agent's issues, then so is any more specific body of information. The formal definition of inquisitive epistemic models is as follows.

DEFINITION 2.1. **Inquisitive Epistemic Model** [8, p. 1650]
An inquisitive epistemic model is a triple $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ where:

- W is the domain of worlds;
- $\mathcal{A}$ is the domain of agents;
- Σ_a is the *inquisitive state map*. It assigns to each world w an inquisitive state $\Sigma_a(w)$, a non-empty downward closed set of information states. These are the states where the agent's issues are resolved;
- $V : \mathcal{P} \to \wp(W)$ is a valuation function that specifies for each atomic formula in which worlds it is true.

The information state of a in world w is $\sigma_a(w) = \bigcup \Sigma_a(w)$: the set of worlds she considers to be candidates for the actual world at w.

In this paper, we restrict ourselves to epistemic cases. That is, we think about knowledge as factive: agents always consider the actual world as one of the possible worlds.

$$\textit{Factivity:}\ \text{for all } w \in W,\ w \in \sigma_a(w)$$

Additionally, in IEL it is assumed that agents are introspective with respect to their knowledge and issues. Formally, this amounts to requiring that the state maps satisfy the following two conditions:

$$\textit{Positive introspection:}\ \text{for all } w, v \in W, \text{ if } v \in \sigma_a(w) \text{ then } \Sigma_a(v) \subseteq \Sigma_a(w)$$

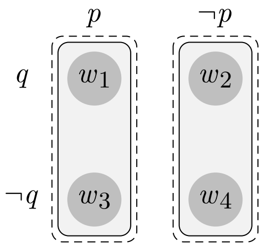
$$\textit{Negative introspection:}\ \text{for all } w, v \in W, \text{ if } v \in \sigma_a(w) \text{ then } \Sigma_a(v) \supseteq \Sigma_a(w)$$

Although it is not necessary for our purposes to require introspection, it does make it easier to represent inquisitive states in diagrams. Therefore, our examples will rely on full introspection, but nothing hinges on this choice.

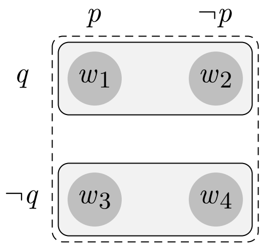
Let us look at an example of an inquisitive epistemic model. Take the following model M with $W = \{w_1, w_2, w_3, w_4\}$, $V(p) = \{w_1, w_3\}$, $V(q) = \{w_1, w_2\}$ and two agents a and b. Suppose that in every world a knows whether p and not whether q, but does not care about it. b does not know whether p or whether q, but wants to know whether q. The model will look like this:[4]

- $\Sigma_a(w_1) = \Sigma_a(w_3) = \{\{w_1, w_3\}\}^{\downarrow}$
- $\Sigma_a(w_2) = \Sigma_a(w_4) = \{\{w_2, w_4\}\}^{\downarrow}$
- $\Sigma_b(w_1) = \Sigma_b(w_2) = \Sigma_b(w_3) = \Sigma_b(w_4) = \{\{w_1, w_2\}, \{w_3, w_4\}\}^{\downarrow}$

Throughout this paper, we will represent inquisitive epistemic models by diagrams. We follow the conventions of [5]: for each world w, the worlds within the same dashed line are the worlds that the agent considers as possible worlds in w. The solid lines represent the issues within each epistemic state: these states and their subsets are the ones that count as resolving the agent's issues. We only draw the maximal elements of the state map. The example model M is represented in Figure 1.



(a) State map for a



(b) State map for b

Figure 1: State maps in an inquisitive epistemic model

Let us now look at the syntax and semantics of IEL. Questions are formed by means of *inquisitive disjunction*: $\varphi \mathbin{\vee\!\!\!\vee} \psi$ is supported in a state just in case it supports one of the disjuncts. The semantics of the knowledge modality K_a generalizes the standard semantics of epistemic logic: this modality can now also operate directly on questions. Furthermore, a new modality E_a is introduced, which allows us to talk about the issues that the agent entertains.

DEFINITION 2.2. **Syntax of $\mathcal{L}^{\mathsf{IEL}}$** [5, p. 254]

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid K_a\varphi \mid E_a\varphi$$

We use the following abbreviations:

Negation:	$\neg\varphi := \varphi \to \bot$
Classical disjunction:	$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
Wonder modality:	$W_a\varphi := \neg K_a\varphi \wedge E_a\varphi$
Polar question:	$?\varphi := \varphi \mathbin{\vee\!\!\!\vee} \neg\varphi$
Tautology:	$\top := \bot \to \bot$

As formulas of IEL are evaluated relative to information states rather than single worlds, we specify the semantics in terms of support rather than truth.

[4] $S^{\downarrow}$ is the *downward closure* of S, the set of all subsets of elements of S.

DEFINITION 2.3. **Support conditions in IEL** [5, p. 47, 50, 255]
Let s be an information state in inquisitive epistemic model M.

$M, s \models p$	iff	$s \subseteq V(p)$
$M, s \models \bot$	iff	$s = \emptyset$
$M, s \models \varphi \wedge \psi$	iff	$s \models \varphi$ and $s \models \psi$
$M, s \models \varphi \to \psi$	iff	for all $t \subseteq s$, $t \models \varphi$ implies $t \models \psi$
$M, s \models \varphi \vee\!\!\!\vee\, \psi$	iff	$s \models \varphi$ or $s \models \psi$
$M, s \models K_a\varphi$	iff	for all $w \in s : M, \sigma_a(w) \models \varphi$
$M, s \models E_a\varphi$	iff	for all $w \in s$, for all $t \in \Sigma_a(w) : M, t \models \varphi$

Truth in a world is defined as support in the corresponding singleton information state.

DEFINITION 2.4. **Truth** [5, p. 48]
Let M be an inquisitive epistemic model and w a world.

$$M, w \models \varphi \iff M, \{w\} \models \varphi$$

This means that the truth conditions for the language are as follows:

FACT 2.1. **Truth conditions in IEL** [5, p. 49, 50, 255]
Let w be a world in inquisitive epistemic model M.

$M, w \models p$	iff	$w \in V(p)$
$M, w \not\models \bot$		
$M, w \models \varphi \wedge \psi$	iff	$w \models \varphi$ and $w \models \psi$
$M, w \models \varphi \to \psi$	iff	$w \not\models \varphi$ or $w \models \psi$
$M, w \models \varphi \vee\!\!\!\vee\, \psi$	iff	$w \models \varphi$ or $w \models \psi$
$M, w \models K_a\varphi$	iff	$M, \sigma_a(w) \models \varphi$
$M, w \models E_a\varphi$	iff	for all $t \in \Sigma_a(w) : M, t \models \varphi$

Inspecting the truth conditions, we notice two things: first, the truth conditions for the propositional fragment are classical. Second, the truth conditions of the modal formulas are dependent on support conditions.

Let us define the support set of φ, which is simply the set of all information states in a model that support φ. Similarly, we define the truth set of φ as the set of worlds in which φ is true.

DEFINITION 2.5. **Support set and truth set** [5, p. 12]
Let φ be a formula and M an inquisitive epistemic model. Then the support set $[\varphi]_M$ is the set of all information states in M where φ is supported:

$$[\varphi]_M = \{s \subseteq W_M \mid M, s \models \varphi\}$$

The truth set $|\varphi|_M$ is the set of all worlds in M in which φ is true:

$$|\varphi|_M = \{w \in W_M \mid M, w \models \varphi\}$$

A fundamental notion in support-conditional semantics is that of truth-conditionality. A formula is truth-conditional just in case its meaning is completely determined by its truth conditions.

DEFINITION 2.6. **Truth-conditionality** [5, p. 26]
A formula φ is *truth-conditional* iff for all models M and states s:

$$M, s \models \varphi \iff M, w \models \varphi \text{ for all } w \in s$$

Following [5], we will regard truth-conditional formulas as statements and all other formulas as questions. As a convention, statements are denoted by α or β and questions by μ or ν.

For any formula φ we can find a statement with the same truth conditions as φ. For instance, since all negations are truth-conditional, the double negation of φ is such a formula.

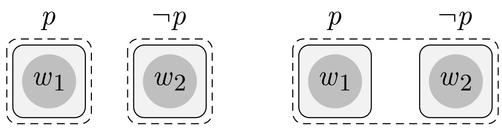
FACT 2.2. **Double negation and truth conditions**
For all $\varphi \in \mathcal{L}^{\mathsf{IEL}}$:

- $\neg\neg\varphi$ is truth-conditional
- $M, w \models \varphi \iff M, w \models \neg\neg\varphi$

2.1.2 Modalities and questions

In IEL, epistemic modalities can range over questions. As we have seen, there are two epistemic modalities, K_a and E_a, and a third one, W_a, defined in terms of the first two. Just as in standard epistemic logic, the K_a-modality expresses knowledge, but its semantics is generalized to questions. For instance, $K_a?p$ is true just in case a knows whether p is the case, that is, if she either knows that p or that $\neg p$. An example is shown in Figure 2(a), where the dashed line indicates that a can distinguish w_1 from w_2. This means that her information state is such that it resolves the question $?p$. Therefore, $K_a?p$ is true in both worlds. The E_a-modality is used to express the issues of agents. For instance, $E_a?p$ means a *entertains* $?p$: she either knows or wants to know the answer to the question $?p$.

The wonder modality, W_a, is an abbreviation of not knowing and entertaining. That is, $W_a\mu$ is true just in case a does not know the answer to the question μ, but does entertain it. An example is shown in Figure 2(b), where the dashed line indicates that w_1 and w_2 are not distinguishable to a. Therefore, her information state does not resolve the question $?p$. However, the solid lines indicate the information states that resolve her issues: $\{w_1\}$ and $\{w_2\}$. Both these information states resolve the question $?p$. Therefore, we say that a wonders $?p$: $W_a?p$. The wonder modality is only meaningful when applied to questions, since $W_a\alpha$ is a contradiction whenever α is a statement.



(a) $K_a?p$ (b) $W_a?p$

Figure 2: Modalities ranging over questions

Because questions can be embedded under epistemic modalities in IEL, we can express not only the knowledge and issues agents have about general facts, but also the knowledge and issues they have about *each other's* knowledge and issues. For instance, that a knows that b wonders whether p: $K_a W_b?p$; or that b wonders whether a knows p: $W_b?K_a p$.

2.1.3 Dynamics

The dynamic variant of IEL, called IDEL [8, 5], extends IEL with public announcements, analogous to the way PAL extends epistemic logic ([19]). The novelty of IDEL is that both questions and statements can be announced. In the former case, the model will be transformed in such a way that agents will typically come to entertain the question. The language $\mathcal{L}^{\mathsf{IDEL}}$ extends $\mathcal{L}^{\mathsf{IEL}}$ with a dynamic modality, $[\varphi]$, which allows us to talk about the situation after the public utterance of φ.

The following definition contains the procedure by which we update a model.

DEFINITION 2.7. **Update in IDEL** [5, p. 311]
Let $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ be an inquisitive epistemic model and $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$. Then the model updated with φ is $M^\varphi = \langle W^\varphi, \{\Sigma_a^\varphi \mid a \in \mathcal{A}\}, V^\varphi\rangle$, defined as follows.

- $W^\varphi = W \cap |\varphi|_M$
- $V^\varphi = V\restriction_{|\varphi|_M}$
- $\Sigma_a^\varphi(w) = \Sigma_a(w) \cap [\varphi]_M$

This definition says that updating an inquisitive epistemic model with a formula φ has two effects. First, the worlds in which φ is false are dropped from the model. Second, for each agent a and world w, the information states that count as resolving the issues of a in w become restricted to the ones that support φ. See Figure 3 for an illustration of the effect of a public utterance of a statement and a question.

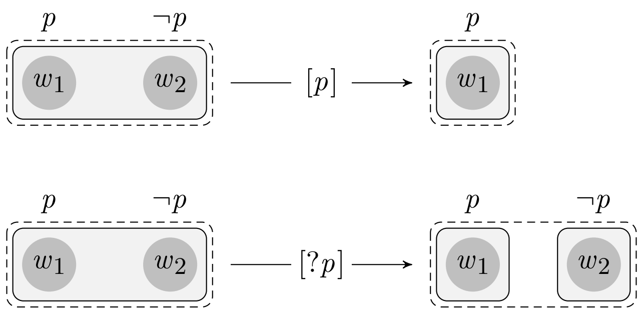


Figure 3: Public utterance in IDEL

The public utterance operator $[\varphi]$ has the following support condition:

DEFINITION 2.8. **Support condition for dynamic modalities** [5, p. 313]

$$M, s \models [\varphi]\psi \iff M^\varphi, s \cap |\varphi|_M \models \psi$$

This means that $[\varphi]\psi$ is supported in information state s of model M just in case ψ is supported in the information state s, restricted to the worlds in which φ is true, in the updated model M^φ. In other words, our information state s is such that it would come to support ψ after a public utterance of φ.

2.2 Action Model Logic

As we mentioned before, utterances in IDEL are always public, whereas in reality agents may not always know for sure what was announced. A common way to encode epistemic events that are not public is by means of *action models* ([3]). These are Kripke structures containing the actions considered possible. What the agents know about these actions is encoded by an accessibility relation. For each action considered, the information that it conveys is given by its *precondition*: a formula that says what has to be true for the action to be executable.

A new epistemic model, which encodes the situation after the epistemic action, is constructed by taking the product update of the original epistemic model and the action model. A formula with a dynamic modality ($[\mathsf{M}, \mathsf{x}]\varphi$) is evaluated in a world w of model M by constructing the product update $M \otimes \mathsf{M}$ and by checking whether if x is executable at w, φ is true in the world $\langle w, \mathsf{x} \rangle$ of this new model.

Our goal is to generalize this action model approach, using IEL instead of standard epistemic logic as our static basis. To achieve this, we need to define a suitable notion of action models and product update. As we will see, action models based on the static logic IEL can be defined in different ways, depending on the interpretation we give to epistemic actions. In standard DEL, two interpretations are common.

One is to think about actions as *announcements*. In this interpretation, an action is a speech act. The accessibility relation in the action model encodes which speech acts are indistinguishable to the agent, and the precondition of each speech act encodes what is being uttered. Enriching action models with questions then amounts to considering not just the speech act of stating, as in AML, but also that of asking. Thus, we can enrich action models by allowing questions as the content of actions.

Under another interpretation of dynamic epistemic logic, actions are not viewed as speech acts but as *observations*.[5] For instance, observations as the result of some scientific experiment. An observation reveals information, which can be expressed by a statement. The action model contains all observations held possible, and the precondition of each observation expresses the information it reveals. The accessibility relation encodes what each agent knows about what is being observed. Adopting this interpretation, we can enrich action models by not just considering the knowledge agents have about which observation is made, but also their issues. For instance, we can think about a scientific experiment in which several researchers are involved, who have different goals.

Note that when there are no questions in the static language, these two different interpretations do not require a different treatment. This is why they are often not explicitly distinguished and used interchangeably.

To summarize, we consider two approaches to the integration of IEL and AML:

- Add *questions* to action models: allow non truth-conditional formulas to serve as the content of actions.

- Add *issues* to action models: change the structure of the action models to encode issues about actions, just like an inquisitive epistemic model encodes issues about worlds.

We work out both approaches in turn, in Sections 3 and 4, respectively.

3 Action Models with Questions

3.1 Definitions

3.1.1 Action models and logical language

We start by defining a new type of action models and the logic this gives rise to, which we will call *Action Model Logic with Questions* (AMLQ).

As we have seen above, in AML the *preconditions* of actions determine what they convey. In this way we can specify the *information* associated with the action. However, in our generalized setting we want to encode actions in which a question is asked, which raises an issue. The content of such actions cannot be determined by a precondition. Hence, while an action's content is completely determined by its precondition in AML, this is not possible in a setting with questions.

[5]Such an interpretation is suggested, for instance, by [17].

Therefore, we make one crucial modification to the original definition of action models from [3]: we separate action content from preconditions. The function $\mathsf{content}$ assigns a formula from our inquisitive logic, $\mathcal{L}^{\mathsf{AMLQ}}$, to each action. We give the following two definitions simultaneously.

DEFINITION 3.1. **Syntax of $\mathcal{L}^{\mathsf{AMLQ}}$**
Let $\mathcal{L}^{\mathsf{AMLQ}_0} = \mathcal{L}^{\mathsf{IEL}}$. For $i > 0$, $\mathcal{L}^{\mathsf{AMLQ}_i}$ is defined as follows, where s is a set of actions within the AMLQ action model M of at most level $i-1$ (defined below).

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

The full language is the union of all $\mathcal{L}^{\mathsf{AMLQ}_i}$ for all natural numbers i.

$$\mathcal{L}^{\mathsf{AMLQ}} := \bigcup_{i \geq 0} \mathcal{L}^{\mathsf{AMLQ}_i}$$

Abbreviations:

- $[\mathsf{s}]\varphi := [\mathsf{M}, \mathsf{s}]\varphi$ (whenever M is clear from the context)
- $[\mathsf{x}]\varphi := [\{\mathsf{x}\}]\varphi$ (where x is a single action)

DEFINITION 3.2. **Action Model with Questions**
For $i > 0$, an AMLQ action model of level i is a triple $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{content}\rangle$, where:

- S is a finite domain of actions;
- For each $a \in \mathcal{A}$, $\sim_a$ is an equivalence relation on S, encoding indistinguishability to agent a;
- $\mathsf{content} : \mathsf{S} \to \mathcal{L}^{\mathsf{AMLQ}_i}$ is a function that assigns a content $\mathsf{content}(\mathsf{x}) \in \mathcal{L}^{\mathsf{AMLQ}_i}$ to each action $\mathsf{x} \in \mathsf{S}$.

For each action x, we denote the set of actions indistinguishable from x to agent a by $\delta_a(\mathsf{x})$. Formally, $\delta_a(\mathsf{x}) = \{\mathsf{y} \mid \mathsf{x} \sim_a \mathsf{y}\}$.

The content of an action can now be a statement or a question. Like in AML, we also assign preconditions to our actions. In case the content is a statement, the precondition is equivalent to the statement itself. Whenever the content is a question, the precondition is a statement which we regard as capturing the question's presupposition that one of the proposed alternatives is true. This means that the precondition of an action is determined by its content. We make use of the general fact that double negation turns any formula into a statement with the same truth conditions.

DEFINITION 3.3. **Precondition of an action**
Given an action x, its precondition $\mathsf{pre}(\mathsf{x})$ is defined as:

$$\mathsf{pre}(\mathsf{x}) := \neg\neg\mathsf{content}(\mathsf{x})$$

Thus, $\mathsf{content}(\mathsf{x})$ expresses what the action conveys, while $\mathsf{pre}(\mathsf{x})$ expresses what has to be true in order to guarantee that the action can be performed. For instance, the precondition of an action with content p is equivalent to p, because it can only be truthfully performed in worlds where p is true. An action with content $?p$ can be truthfully performed in any world, as it contains no information. Therefore its precondition is $\neg\neg ?p \equiv \top$. We have the following proposition, which will play an important part in the rest of our proofs.

PROPOSITION 3.1. **Action content and precondition have equal truth conditions**
For all actions x, for every inquisitive epistemic model M and world w:

$$M, w \models \mathsf{content}(\mathsf{x}) \iff M, w \models \mathsf{pre}(\mathsf{x})$$

Proof. Immediate from Definition 3.3 and Fact 2.2. □

3.1.2 Update procedure

We define a procedure that describes how to compute the product update of an IEL model and an AMLQ action model. This is an IEL model that encodes the updated knowledge *and issues* of the agents.

We first introduce two projection operators, that allow us to associate to each state s in the updated model a state $\pi_1(s)$ in the original model and a state $\pi_2(s)$ in the action model.

DEFINITION 3.4. **Projection operators**
If $s \subseteq W \times \mathsf{S}$, then:

$$\pi_1(s) := \{w \mid \langle w, \mathsf{x}\rangle \in s \text{ for some } \mathsf{x}\}$$

$$\pi_2(s) := \{\mathsf{x} \mid \langle w, \mathsf{x}\rangle \in s \text{ for some } w\}$$

Like in standard action model logic, the domain of the updated model is the cartesian product of that of the original model and the action model, restricted to pairs of which the world satisfies the precondition of the action.

The crucial part of the definition is that of the updated state map. We formulate four conditions for an information state s to belong to the inquisitive state of some agent in a world $\langle w, \mathsf{x}\rangle$. These conditions are as follows:

(i) The knowledge and issues an agent had before the action have to be preserved. Hence, a state can only count as resolving the agent's new issues if it resolves her old issues. This means that $\pi_1(s) \in \Sigma_a(w)$;

(ii) The agent's knowledge about the action should be preserved. Thus, the actions associated with this state have to be indistinguishable from x. Formally, $\pi_2(s) \subseteq \delta_a(\mathsf{x})$;

(iii) The state specifies exactly which action has occurred (we assume agents at least *want* to know this). This means that $\pi_2(s)$ contains exactly one action y, or it is empty (in case s is the empty state);

(iv) The information state should support the content of the action it specifies. This is trivial for statements, but not for questions, since in this case an information state is not necessarily informative enough to resolve the question it specifies.

This requirement has the effect that agents will typically come to entertain a question if they are sure that it was asked.[6] Formally, we require that $M, \pi_1(s) \models \mathsf{content}(\mathsf{y})$, where y is the unique action in $\pi_2(s)$, if present.

[6] This is not always the case: because questions can be about the knowledge and issues of agents, they might have a presupposition that becomes false once uttered.

DEFINITION 3.5. **Product update**
Let M be an inquisitive epistemic model and M an AMLQ action model. Then $M' = (M \otimes \mathsf{M})$ is the product update of M and M, defined as follows.

$M' = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V' \rangle$, where:

- $W' = \{\langle w, \mathsf{x}\rangle \mid w \in W, \mathsf{x} \in \mathsf{S} \text{ and } M, w \models \mathsf{pre}(\mathsf{x})\}$
- $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ iff
  (i) $\pi_1(s) \in \Sigma_a(w)$
  (ii) $\pi_2(s) \subseteq \delta_a(\mathsf{x})$
  (iii) There is at most one $\mathsf{y} \in \pi_2(s)$
  (iv) $\forall \mathsf{y} \in \pi_2(s) : M, \pi_1(s) \models \mathsf{content}(\mathsf{y})$
- $\langle w, \mathsf{x}\rangle \in V'(p)$ iff $w \in V(p)$

3.1.3 Semantics

Notice that we have extended IEL with dynamic modalities for sets of actions s. We interpret these set modalities as expressing information about which action is executed: it is one action $\mathsf{x} \in \mathsf{s}$, but it is not determined which one. We then recover the execution of a single action as a special case, namely the case where s is a singleton.

The semantics of AMLQ consists of the support conditions of IEL, extended with a new support condition for the dynamic modalities. Before we can define this support condition, we need a way to connect states in the original model to the corresponding states in the updated model. Let M be an inquisitive epistemic model and s an information state in that model. Let M be an AMLQ action model and s a set of actions in that model. Furthermore, let $M' = M \otimes \mathsf{M}$.

DEFINITION 3.6. **Updated state**
$s[\mathsf{M}, \mathsf{s}]$ is the information state in M' such that:

$$s[\mathsf{M}, \mathsf{s}] = \{\langle w, \mathsf{x}\rangle \in W' \mid w \in s \text{ and } \mathsf{x} \in \mathsf{s}\}$$

We allow omission of the action model M in the notation of updated states. Notice that by the above definition, the set $s[\mathsf{s}]$ consists of all the pairs $\langle w, \mathsf{x}\rangle \in s \times \mathsf{s}$ such that $M, w \models \mathsf{pre}(\mathsf{x})$. Using the notion of updated states, we can now give the support condition for dynamic modalities.

DEFINITION 3.7. **Support condition for dynamic modalities**
The support condition for dynamic modalities is the following:

$$M, s \models [\mathsf{M}, \mathsf{s}]\varphi \iff M', s[\mathsf{M}, \mathsf{s}] \models \varphi$$

Like in IEL, truth in a world is defined as support in the corresponding singleton state. We therefore have the following truth condition for dynamic modalities.

FACT 3.1. **Truth condition for dynamic modalities**
Let $w[\mathsf{M}, \mathsf{s}] = \{\langle w, \mathsf{x}\rangle \in W' \mid \mathsf{x} \in \mathsf{s}\}$. Then the truth condition for dynamic modalities is the following:

$$M, w \models [\mathsf{M}, \mathsf{s}]\varphi \iff M', w[\mathsf{M}, \mathsf{s}] \models \varphi$$

Because s may contain multiple actions, $w[\mathsf{s}]$ may be an information state consisting of more than one world. This means that, for $[\mathsf{s}]\varphi$ to be *true* in w, φ must be *supported* in $w[\mathsf{s}]$. If we want to give a characterization of truth of dynamic formulas only in terms of truth, we can only do this for dynamic modalities where the action is determined:

$$M, w \models [\mathsf{M}, \mathsf{x}]\varphi \iff w \models \mathsf{pre}(\mathsf{x}) \text{ implies } M', \langle w, \mathsf{x}\rangle \models \varphi$$

This formulation makes it clear that $[\mathsf{x}]\varphi$ is true in w just in case either the action x is incompatible with w (making the formula vacuously true) or φ is true in the corresponding world $\langle w, \mathsf{x}\rangle$ in the product update. This corresponds exactly to the truth condition of dynamic modalities in AML.

The intuitive reading of a formula with a dynamic modality, like $[\mathsf{x}]\varphi$, should be: 'φ is supported after action x is executed'. A formula with a dynamic modality that contains a set s of actions, like $[\mathsf{s}]\varphi$, can be read as: 'after getting the information that *some action* of s is executed, φ is supported'. This means we think of sets of actions s in the same way as we think of information states s. Namely, as encoding the information that the actual world (action) is one of the worlds (actions) in the set.

3.1.4 Epistemic states and state maps in updated models

The following lemma tells us that the information state of a in world $\langle w, \mathsf{x}\rangle$ of the updated model is simply the information state of a in world w of the original model updated with the set of actions which a considers possible, $\delta_a(\mathsf{x})$.

LEMMA 3.1. **Epistemic states in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in action model M and $M' = M \otimes \mathsf{M}$. Let $\langle w, \mathsf{x}\rangle$ be a world in M'. Then we have the following:

$$\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$$

Proof. By unpacking Definition 3.5 and Definition 3.6. □

This shows that epistemic states in the updated model are obtained as in standard action model logic. The next lemma shows how inquisitive states in the original and updated models are related.

LEMMA 3.2. **State maps in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in action model M and $M' = M \otimes \mathsf{M}$. Suppose $\langle w, \mathsf{x}\rangle \in W'$. Then we have the following:

$$\Sigma'_a(\langle w, \mathsf{x}\rangle) = \{s[\mathsf{y}] \mid s \in \Sigma_a(w), \mathsf{y} \sim_a \mathsf{x} \text{ and } M, s \models \mathsf{content}(\mathsf{y})\}$$

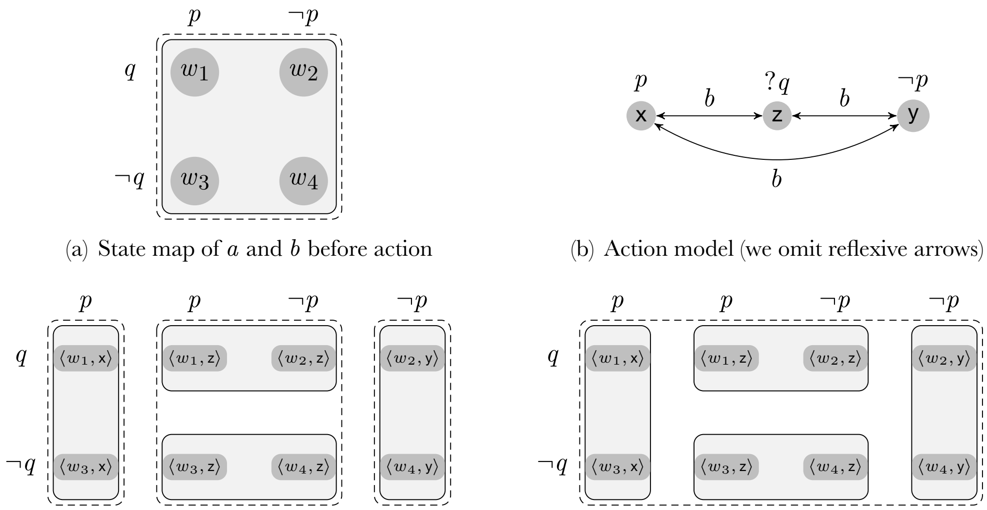
Proof. By unpacking Definition 3.5. □

The above lemma may be viewed as an alternative formulation of the definition of Σ'_a.

3.2 Example

Let us now look at a simple example to see the mechanisms at work. Suppose Anna and Bob have invited their friends Peter and Quinn to their party. Initially, both Anna (a) and Bob (b) have no knowledge or issues about whether Peter is attending (p) or Quinn is attending (q), as illustrated in Figure 4(a).

Then, Anna gets a call from Peter. He asks whether Quinn is attending ($?q$). However, Bob considers it possible that Peter calls to say whether he himself is attending (p) or not ($\neg p$). We encode



(a) State map of a and b before action

(b) Action model (we omit reflexive arrows)

(c) State map of a after action

(d) State map of b after action

Figure 4: Example

this epistemic action in an action model with three actions: x, y and z, where $\mathsf{content}(\mathsf{x}) = p$, $\mathsf{content}(\mathsf{y}) = \neg p$ and $\mathsf{content}(\mathsf{z}) = ?q$ (Figure 4(b)).

We calculate the product update (Figure 4(c) and 4(d)). The actual world in the updated model is one of the z-worlds. Thus, after the phone call, Anna entertains the issue whether Quinn is attending (all information states in the original model support $[\mathsf{z}]E_a?q$), but Bob does not ($[\mathsf{z}]\neg E_b?q$).

3.3 Properties of AMLQ

In this section we introduce some interesting properties of AMLQ, which we will later use in proofs.

3.3.1 Persistence and empty state

We start by noting that the support relation of AMLQ has the persistence property and the empty state property, which are the hallmark of inquisitive logics.

PROPOSITION 3.2. **Properties of the support relation**
For all models M and formulas $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, we have the following properties:

- *Persistence property:* if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$.
- *Empty state property:* $\emptyset \models \varphi$

Proof. By unpacking the support conditions. □

The persistence property corresponds to the intuitive idea that if some body of information is enough to support a formula, than so is any more specific body of information. The empty set is interpreted as the *inconsistent state* and thus supports every formula.

3.3.2 Declaratives

As in IEL, we can syntactically define a fragment of our language, the *declaratives*, and prove that all of its members are truth-conditional.

DEFINITION 3.8. **Declarative fragment of $\mathcal{L}^{\mathsf{AMLQ}}$**
The set of declarative formulas $\mathcal{L}_!^{\mathsf{AMLQ}}$ is defined inductively as follows, where $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{s}]\alpha$$

Although $K_a\varphi$ and $E_a\varphi$ are truth-conditional even if φ is not, this is not the case for $[\mathsf{s}]\varphi$, as can be seen from the support condition of the dynamic modality. Therefore, we say that $[\mathsf{s}]\alpha$ is a declarative just in case α is.

PROPOSITION 3.3. Any $\alpha \in \mathcal{L}_!^{\mathsf{AMLQ}}$ is truth-conditional

Proof. By induction on the complexity of α. All steps of the proof proceed as in IEL ([5, p. 260]). We only add the step for the dynamic modality. By the induction hypothesis, α is truth-conditional. Then we use the support condition of the dynamic modality and the definition of a state in an updated model to obtain:

$$\begin{aligned}
M, s \models [\mathsf{s}]\alpha &\iff M', s[\mathsf{s}] \models \alpha \\
&\iff \text{for all } \langle w, \mathsf{x}\rangle \in s[\mathsf{s}] : M', \langle w, \mathsf{x}\rangle \models \alpha \\
&\iff \text{for all } \langle w, \mathsf{x}\rangle \in W' \text{ such that } w \in s \text{ and } \mathsf{x} \in \mathsf{s} : M', \langle w, \mathsf{x}\rangle \models \alpha \\
&\iff \text{for all } w \in s \text{ such that } \langle w, \mathsf{x}\rangle \in W' \text{ and } \mathsf{x} \in \mathsf{s} : M', \langle w, \mathsf{x}\rangle \models \alpha \\
&\iff \text{for all } w \in s : M', w[\mathsf{s}] \models \alpha \\
&\iff \text{for all } w \in s : M, w \models [\mathsf{s}]\alpha
\end{aligned}$$

By Definition 2.6, $[\mathsf{s}]\alpha$ is truth-conditional. □

3.3.3 Resolutions and normal form

An important result, familiar from IEL, is that every formula of $\mathcal{L}^{\mathsf{AMLQ}}$ is equivalent to an inquisitive disjunction of declarative formulas, namely its *resolutions*.

DEFINITION 3.9. **Resolutions** (based on [5, p. 261])
For any formula $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, its set of resolutions $\mathcal{R}(\varphi)$ is defined inductively as follows:

- $\mathcal{R}(\alpha) = \{\alpha\}$ if α is an atom, $\bot$ or a modal formula $K_a\varphi$ or $E_a\varphi$
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f \text{ is a function from } \mathcal{R}(\varphi) \text{ to } \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$
- $\mathcal{R}([\mathsf{s}]\varphi) = \{[\mathsf{s}]\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$

PROPOSITION 3.4. **A formula is supported iff some resolution of it is**
For all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, for every inquisitive epistemic model M and state s:

$$M, s \models \varphi \iff M, s \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

Proof. By induction on the complexity of φ. All steps of the proof proceed as in IEL ([5, p. 261]). We only add the step for the dynamic modality. By the induction hypothesis, $M, s \models \varphi \iff M, s \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$. Then we use the support condition of the dynamic modality and Definition 3.9 to obtain:

$$\begin{aligned} M, s \models [\mathsf{s}]\varphi &\iff M', s[\mathsf{s}] \models \varphi \\ &\iff M', s[\mathsf{s}] \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi) \\ &\iff M, s \models [\mathsf{s}]\alpha \text{ for some } \alpha \in \mathcal{R}(\varphi) \\ &\iff M, s \models \alpha \text{ for some } \alpha \in \mathcal{R}([\mathsf{s}]\varphi) \end{aligned}$$

□

PROPOSITION 3.5. **Normal form**
For all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, $\varphi \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.

Proof. Resolution sets are by definition finite. Hence, this proposition follows immediately from Proposition 3.4 and the support condition of inquisitive disjunction. □

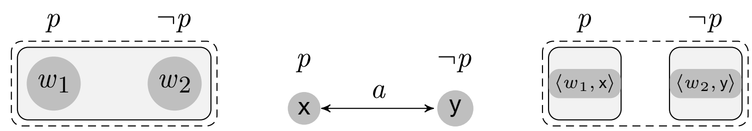
3.3.4 Set modalities and non-deterministic actions

Before we continue, let us point out some differences between set modalities in AMLQ and non-deterministic action modalities in AML.

In AML there are dynamic modalities not just for simple actions, but also for complex actions, e.g. we have formulas of the form $[\mathsf{x} \cup \mathsf{y}]\alpha$. Like in PDL [9], the action $\mathsf{x} \cup \mathsf{y}$ is taken to be the single action of non-deterministically executing the action x or y. This seems very similar to $[\{\mathsf{x}, \mathsf{y}\}]\alpha$ in AMLQ, which we take to mean that our information implies that α is true after getting the information that either of x and y is executed.

These two interpretations of dynamic modalities are in fact interchangeable, as long as the language does not contain questions. Thus, the informational interpretation we give to these modalities is also available in AML. However, the two interpretations come apart when questions are around. If μ is a question, $[\mathsf{s}]\mu$ is stronger than just stating that after any $\mathsf{x} \in \mathsf{s}$, μ is supported. To see this, consider the example depicted in Figure 5.

The original information state of agent a supports both $[\mathsf{x}]p$ and $[\mathsf{y}]\neg p$, as can be seen from Figure 5(c): given the information that x is occurring, it supports p, while given the information that y is occurring, it supports $\neg p$. So, after any action in the set $\{\mathsf{x}, \mathsf{y}\}$, $?p$ is supported. However, a's original information state does not support $[\{\mathsf{x}, \mathsf{y}\}]?p$: the information that one of x and y is executed is not enough to settle $?p$, since the information state in the updated model that corresponds with this information ($\{\langle w_1, \mathsf{x}\rangle, \langle w_2, \mathsf{y}\rangle\}$) does not support $?p$.



(a) State map of a before action (b) Action model (c) State map of a after action

Figure 5: Example to show that $[\{\mathsf{x}, \mathsf{y}\}]\varphi \not\equiv [\mathsf{x}]\varphi \wedge [\mathsf{y}]\varphi$

This shows the following:

PROPOSITION 3.6. $[\{\mathsf{x}, \mathsf{y}\}]\varphi \not\equiv [\mathsf{x}]\varphi \wedge [\mathsf{y}]\varphi$

This is a fundamental difference between dynamic modalities in support-conditional semantics and in truth-conditional semantics. In logics based on the latter, like AML, this equivalence does hold [23, p. 152]. As one would expect, this equivalence does hold in AMLQ whenever φ is truth-conditional. We can even be a bit more general and prove this for all sets s.[7]

PROPOSITION 3.7. If α is truth-conditional, then $[\mathsf{s}]\alpha \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha$

Proof. As both formulas are declaratives, they are truth-conditional. This means that in order to prove the equivalence, we only need to show that they have the same truth conditions. From the fact that α is truth-conditional, the definition of $s[\mathsf{s}]$ and the support conditions of the dynamic modality and conjunction we can obtain:

$$\begin{aligned} M, w \models [\mathsf{s}]\alpha &\iff M', w[\mathsf{s}] \models \alpha \\ &\iff \text{for all } \langle w, \mathsf{x}\rangle \in w[\mathsf{s}] : M', \langle w, \mathsf{x}\rangle \models \alpha \\ &\iff \text{for all } \mathsf{x} \in \mathsf{s} : \text{if } \langle w, \mathsf{x}\rangle \in W', \text{ then } M', \langle w, \mathsf{x}\rangle \models \alpha \\ &\iff \text{for all } \mathsf{x} \in \mathsf{s} : M', w[\mathsf{x}] \models \alpha \\ &\iff \text{for all } \mathsf{x} \in \mathsf{s} : M, w \models [\mathsf{x}]\alpha \\ &\iff M, w \models \textstyle\bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha \end{aligned}$$

□

3.4 Reduction

We can axiomatize AMLQ using the same strategy as is used for AML ([23]) and IDEL ([5]), namely by showing that the dynamic logic is not more expressive than the static logic it extends (in this case, IEL).

3.4.1 Atom and $\bot$

We start by showing that, like in AML, the following equivalences hold for formulas in which a dynamic modality precedes an atom or $\bot$. As usual, x denotes a single action.

PROPOSITION 3.8. $[\mathsf{x}]p \equiv \mathsf{pre}(\mathsf{x}) \to p$

Proof. As both formulas are declaratives, we only need to show that they have the same truth conditions. We can show this using the truth condition of the dynamic modality, the definition of a state in an updated model and the truth condition of implication.

$$\begin{aligned} M, w \models [\mathsf{x}]p &\iff M', w[\mathsf{x}] \models p \\ &\iff w[\mathsf{x}] = \emptyset \text{ or } M', \langle w, \mathsf{x}\rangle \models p \\ &\iff M, w \not\models \mathsf{pre}(\mathsf{x}) \text{ or } M, w \models p \\ &\iff M, w \models \mathsf{pre}(\mathsf{x}) \to p \end{aligned}$$

□

In a similar way we can obtain a reduction equivalence for $\bot$.

PROPOSITION 3.9. $[\mathsf{x}]\bot \equiv \neg\mathsf{pre}(\mathsf{x})$

Proof. Again both formulas are declaratives, so we can show this with a proof similar to the previous one. □

[7] Notice that s can be empty. In that case, we have an empty conjunction, which we take to be $\top$.

3.4.2 Conjunction and inquisitive disjunction

Whenever a dynamic modality precedes a conjunction, we can distribute it over the conjunction, just like in AML and IDEL. This goes for dynamic modalities of single actions as well as sets.

PROPOSITION 3.10. $[\mathsf{s}](\varphi \wedge \psi) \equiv [\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi$

Proof. By unpacking the support conditions of the dynamic modality and conjunction. □

With dynamic modalities that precede an inquisitive disjunction, we can do the same.

PROPOSITION 3.11. $[\mathsf{s}](\varphi \vee\!\!\!\vee \psi) \equiv [\mathsf{s}]\varphi \vee\!\!\!\vee [\mathsf{s}]\psi$

Proof. Analogous to Proposition 3.10. □

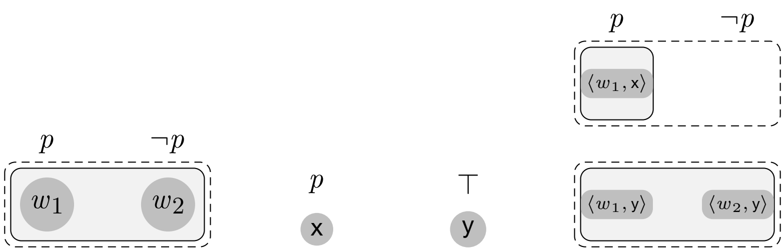
3.4.3 Implication

Although one might expect that we can have a similar reduction equivalence for implication, this is in fact not the case, as the following proposition shows.

PROPOSITION 3.12. $[\mathsf{s}](\varphi \to \psi) \not\equiv [\mathsf{s}]\varphi \to [\mathsf{s}]\psi$

Proof. Consider the original state map, action model and updated state map in Figure 6. Let $s = \{w_1, w_2\}$ and $\mathsf{s} = \{\mathsf{x}, \mathsf{y}\}$. Then $s[\mathsf{s}]$ is the information state consisting of all the worlds in the updated model. This information state does not support $\neg K_a p$, because it contains a world $\langle w_1, \mathsf{x}\rangle$ in which a does know that p. So $s[\mathsf{s}] \not\models K_a p \to \bot$, which in turn means that $s \not\models [\mathsf{s}](K_a p \to \bot)$.

However, the only subset of s that supports $[\mathsf{s}]K_a p$ is the empty set, and the empty set supports $[\mathsf{s}]\bot$. Thus, $s \models [\mathsf{s}]K_a p \to [\mathsf{s}]\bot$. □



(a) State map of a before action

(b) Action model

(c) State map of a after action

Figure 6: Example to show that $[\mathsf{s}](\varphi \to \psi) \not\equiv [\mathsf{s}]\varphi \to [\mathsf{s}]\psi$

The problem with the right to left direction is that to show from any assumption that a state s supports $[\mathsf{s}](\varphi \to \psi)$, we need to show that any $t' \subseteq s[\mathsf{s}]$ that supports φ, supports ψ as well. However, this t' may be any subset, meaning that t' may not be equal to $t[\mathsf{t}]$ for any t and t. Therefore, there is no way we can use the support conditions of the dynamic modality and implication to find a reduction equivalence for $[\mathsf{s}](\varphi \to \psi)$.

This problem does not occur when s is a singleton: any subset of $s[\mathsf{x}]$ is in fact equal to some $t[\mathsf{x}]$ where $t \subseteq s$. We can use this to show the following equivalence.

PROPOSITION 3.13. $[\mathsf{x}](\varphi \to \psi) \equiv [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$

Proof.

($\Rightarrow$) Assume $M, s \models [\mathsf{x}](\varphi \to \psi)$.

Then by the support condition of the dynamic modality, $M', s[\mathsf{x}] \models \varphi \to \psi$. Take any $t \subseteq s$ such that $M, t \models [\mathsf{x}]\varphi$. Then $M', t[\mathsf{x}] \models \varphi$. As $t[\mathsf{x}]$ is a subset of $s[\mathsf{x}]$, $M', t[\mathsf{x}] \models \psi$ by the support condition of implication. This means that $M, t \models [\mathsf{x}]\psi$. By the support condition of implication we have $M, s \models [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$.

($\Leftarrow$) Assume $M, s \models [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$.

Take any $t' \subseteq s[\mathsf{x}]$ such that $t' \models \varphi$. Let $t = \pi_1(t')$. Then by the definition of updated states, $t[\mathsf{x}] = t'$. As $M', t[\mathsf{x}] \models \varphi$, $M, t \models [\mathsf{x}]\varphi$. By the support condition of implication, $M, t \models [\mathsf{x}]\psi$. So $M', t[\mathsf{x}] \models \psi$. As t' was an arbitrary subset of $s[\mathsf{x}]$, by the support condition of implication, $M', s[\mathsf{x}] \models \varphi \to \psi$. Therefore $M, s \models [\mathsf{x}](\varphi \to \psi)$. □

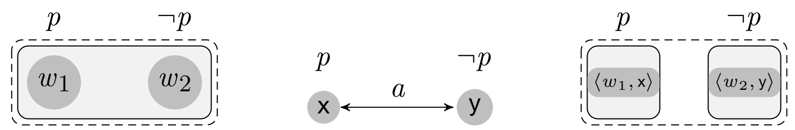
As we will see, this equivalence suffices to guarantee that any formula of AMLQ is equivalent to a static formula of IEL. However, it will pose a challenge for the proof.

3.4.4 Knowledge modality

While in AML, $[\mathsf{x}]K_a\varphi$ is equivalent to $\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]\varphi$, this is not the case in AMLQ. The reason for this is that in AMLQ, φ can be a question. Indeed, for statements α, knowing α after x is (given that the precondition of x is met) the same as knowing that after any action indistinguishable from x, α will be the case. However, this does not generalize to questions.

PROPOSITION 3.14. $[\mathsf{x}]K_a\varphi \not\equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]\varphi$

Proof. Consider the following counterexample, illustrated in Figure 7: agent a has no knowledge or issues in the original model and no knowledge about which action is taking place, which is in fact action x.



(a) State map of a before action
(b) Action model
(c) State map of a after action

Figure 7: Example to show that $[\mathsf{x}]K_a\varphi \not\equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]\varphi$

Clearly, in the updated model, a still does not know whether p is true or false. So $[\mathsf{x}]K_a?p$ is false in w_1. However, $\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]?p$ is true in w_1, because a does know that $[\mathsf{x}]p$ and $[\mathsf{y}]\neg p$, which together make the consequent true. □

Since φ can be a question in our setting, knowing φ after some action x is not the same as knowing that after y, φ, for each action y indistinguishable from x.

In fact, for $[\mathsf{x}]K_a?p$ to be true, it is required that either $K_a([\mathsf{x}]p \wedge [\mathsf{y}]p)$ or $K_a([\mathsf{x}]\neg p \wedge [\mathsf{y}]\neg p)$. This means we could come up with an equivalence for $[\mathsf{x}]K_a\varphi$ by quantifying over the *resolutions* of φ. However, we get a more concise equivalence if we use the set modality to express the uncertainty about which action is the actual one. We can show that after x, a knows φ just in case $\mathsf{pre}(\mathsf{x})$ is false or a's current information state is such that learning that one of $\delta_a(\mathsf{x})$ is executed is enough to settle φ.

PROPOSITION 3.15. $[\mathsf{x}]K_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \rightarrow K_a[\delta_a(\mathsf{x})]\varphi$

Proof. As both formulas are declaratives, we only need to show that they have the same truth conditions.

($\Rightarrow$) Assume $M, w \models [\mathsf{x}]K_a\varphi$.

Then $M', w[\mathsf{x}] \models K_a\varphi$ by the truth condition of the dynamic modality. Assume $M, w \models \mathsf{pre}(\mathsf{x})$. By the truth condition of the knowledge modality, $M', \sigma'_a(\langle w, \mathsf{x}\rangle) \models \varphi$.

By Lemma 3.1, $\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$. So $M', \sigma_a(w)[\delta_a(\mathsf{x})] \models \varphi$. We use the truth condition of the dynamic modality to obtain $M, \sigma_a(w) \models [\delta_a(\mathsf{x})]\varphi$. By the truth condition of the knowledge modality, $M, w \models K_a[\delta_a(\mathsf{x})]\varphi$. We can then drop our assumption that $M, w \models \mathsf{pre}(\mathsf{x})$ to obtain $M, w \models \mathsf{pre}(\mathsf{x}) \rightarrow K_a[\delta_a(\mathsf{x})]\varphi$.

($\Leftarrow$) Assume $M, w \models \mathsf{pre}(\mathsf{x}) \rightarrow K_a[\delta_a(\mathsf{x})]\varphi$.

Either $M, w \models \mathsf{pre}(\mathsf{x})$ or $M, w \not\models \mathsf{pre}(\mathsf{x})$. In the latter case, we immediately have $M, w \models [\mathsf{x}]K_a\varphi$, since $w[\mathsf{x}] = \emptyset$. In the former case, we have $M, w \models K_a[\delta_a(\mathsf{x})]\varphi$. By the truth condition of the knowledge modality, $M, \sigma_a(w) \models [\delta_a(\mathsf{x})]\varphi$. Then by the truth condition the dynamic modality, $M', \sigma_a(w)[\delta_a(\mathsf{x})] \models \varphi$. Since by Lemma 3.1, $\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$, that means $M', \sigma'_a(\langle w, \mathsf{x}\rangle) \models \varphi$. Hence, $M', w[\mathsf{x}] \models K_a\varphi$. By the truth condition of the dynamic modality, $M, w \models [\mathsf{x}]K_a\varphi$. □

3.4.5 Entertain modality

We now move on to the reduction equivalence for the entertain modality.

PROPOSITION 3.16. $[\mathsf{x}]E_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \rightarrow \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \rightarrow [\mathsf{y}]\varphi)$

Proof. As both formulas are declaratives, we only need to show that they have the same truth conditions.

($\Rightarrow$) Assume $M, w \models [\mathsf{x}]E_a\varphi$.

Then $M', w[\mathsf{x}] \models E_a\varphi$. Assume $M, w \models \mathsf{pre}(\mathsf{x})$. By the truth condition of the entertain modality, we have for all $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle) : M', s' \models \varphi$.

Take any action z such that $\mathsf{z} \sim_a \mathsf{x}$ and some $s \in \Sigma_a(w)$. Then take any $t \subseteq s$ such that $M, t \models \mathsf{content}(\mathsf{z})$. By Lemma 3.2, $t[\mathsf{z}] \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. This means that $M', t[\mathsf{z}] \models \varphi$.

By the support condition of the dynamic modality, we have $M, t \models [\mathsf{z}]\varphi$. By the support condition of implication, we have $M, s \models \mathsf{content}(\mathsf{z}) \rightarrow [\mathsf{z}]\varphi$. As s was an arbitrary state in $\Sigma_a(w)$, we have $M, w \models E_a(\mathsf{content}(\mathsf{z}) \rightarrow [\mathsf{z}]\varphi)$. As z was an arbitrary action indistinguishable by a from x, we have $M, w \models \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \rightarrow [\mathsf{y}]\varphi)$. Then, finally, we drop our assumption that $M, w \models \mathsf{pre}(\mathsf{x})$ to obtain $M, w \models \mathsf{pre}(\mathsf{x}) \rightarrow \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \rightarrow [\mathsf{y}]\varphi)$.

($\Leftarrow$) Assume $M, w \models \mathsf{pre}(\mathsf{x}) \rightarrow \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \rightarrow [\mathsf{y}]\varphi)$.

Either $M, w \models \mathsf{pre}(\mathsf{x})$ or $M, w \not\models \mathsf{pre}(\mathsf{x})$. In the latter case, we immediately have $M, w \models [\mathsf{x}]E_a\varphi$, since $w[\mathsf{x}] = \emptyset$ and we are done, so assume the former. Then we have $M, w \models \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \rightarrow [\mathsf{y}]\varphi)$.

As $M, w \models \mathsf{pre}(\mathsf{x})$, we have a world $\langle w, \mathsf{x}\rangle$ in the updated model. Take any $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. Then by Lemma 3.2, $s' = s[\mathsf{z}]$ for some s, z such that $s \in \Sigma_a(w)$, $\mathsf{z} \sim_a \mathsf{x}$ and $M, s \models \mathsf{content}(\mathsf{z})$. It follows from $\mathsf{z} \sim_a \mathsf{x}$ that $M, w \models E_a(\mathsf{content}(\mathsf{z}) \to [\mathsf{z}]\varphi)$. Then by $s \in \Sigma_a(w)$ we have that $M, s \models \mathsf{content}(\mathsf{z}) \to [\mathsf{z}]\varphi$. As $M, s \models \mathsf{content}(\mathsf{z})$, we obtain $M, s \models [\mathsf{z}]\varphi$. By the support condition of the dynamic modality, $M', s' \models \varphi$.

As s' was an arbitrary state in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$, we have $M', \langle w, \mathsf{x}\rangle \models E_a\varphi$, which means that $M', w[\mathsf{x}] \models E_a\varphi$. By the truth condition of the dynamic modality, we have $M, w \models [\mathsf{x}]E_a\varphi$. □

Since any formula of the form $K_a\varphi$ or $E_a\varphi$ is a declarative, the reduction equivalences for $[\mathsf{x}]K_a\varphi$ and $[\mathsf{x}]E_a\varphi$ can be combined with Proposition 3.7 to reduce formulas of the form $[\mathsf{s}]K_a\varphi$ and $[\mathsf{s}]E_a\varphi$. We now have all the reduction equivalences we need to prove the following theorem.

THEOREM 3.1. **Every formula of AMLQ is equivalent to some formula of IEL**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$.

Proof sketch. The challenge for the proof is the lack of a reduction equivalence for formulas of the form $[\mathsf{s}](\varphi \to \psi)$, where s consists of more than one action and $\varphi \to \psi$ is a question, since neither Proposition 3.7 nor Proposition 3.13 can be used in this case.

However, we can use the fact that $[\mathsf{s}](\varphi \to \psi)$ is equivalent to its normal form: the inquisitive disjunction of its resolutions. The resolutions of $[\mathsf{s}](\varphi \to \psi)$ are guaranteed to be declaratives, which means they can be reduced using Proposition 3.7. The full proof is given in Appendix A.

3.5 Axiomatizing AMLQ

A sound and complete proof system for AMLQ can be obtained by combining all the inference rules for IEL [4] and the rules in Figure 8. The rules in this figure are reduction rules that correspond to the equivalences we proved in Section 3.4, together with a rule of replacement of equivalents. The relation of derivability in this system is denoted by $\vdash$ and the relation of inter-derivability by $\dashv\vdash$.

THEOREM 3.2. **AMLQ is sound and complete**
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{AMLQ}}$, $\Phi \models \psi \iff \Phi \vdash \psi$.

The proof is given in Appendix A.

4 Action Models with Issues

We now turn to the second approach of merging IEL and AML. Instead of adding questions to action models, we will now interpret actions as possible observations and add *issues* as to what is observed.

We do this by giving action models the structure of inquisitive epistemic models: we equip each agent in the action model with an inquisitive state map. This state map assigns to each action an inquisitive state: the set of information states (in this case, sets of actions) that count as resolving the agent's issue with respect to the observations. Thus, the state map does not just encode the indistinguishability of actions (uncertainty about what is observed) but also the agent's issues about what is being observed.

Like in the previous section, we will also introduce a new logical language, which extends IEL with dynamic modalities. We will refer to it as *Action Model Logic with Issues* (AMLI).

!Atom
$$\frac{[\mathsf{x}]p}{\mathsf{pre}(\mathsf{x}) \to p}$$

!$\wedge$
$$\frac{[\mathsf{s}](\varphi \wedge \psi)}{[\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi}$$

!K
$$\frac{[\mathsf{x}]K_a\varphi}{\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi}$$

!$\bot$
$$\frac{[\mathsf{x}]\bot}{\neg\mathsf{pre}(\mathsf{x})}$$

!$\to$
$$\frac{[\mathsf{x}](\varphi \to \psi)}{[\mathsf{x}]\varphi \to [\mathsf{x}]\psi}$$

!E
$$\frac{[\mathsf{x}]E_a\varphi}{\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \to [\mathsf{y}]\varphi)}$$

AUD
$$\frac{[\mathsf{s}]\alpha}{\bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha}$$

!$\vee\!\!\!\vee$
$$\frac{[\mathsf{s}](\varphi \vee\!\!\!\vee \psi)}{[\mathsf{s}]\varphi \vee\!\!\!\vee [\mathsf{s}]\psi}$$

RE
$$\frac{\varphi \leftrightarrow \psi}{\chi[\varphi/p] \leftrightarrow \chi[\psi/p]}$$

Figure 8: The inference rules for dynamic modalities in AMLQ. The double lines indicate that the inference is allowed in both directions. The rule AUD (action uncertainty distribution) can only be applied to declaratives α.

4.1 Definitions

The contents of actions will be formulas of $\mathcal{L}_!^{\mathsf{AMLI}}$, the *declarative fragment* of $\mathcal{L}^{\mathsf{AMLI}}$, defined as in Definition 3.8. The syntax of the language is defined analogous to Definition 3.1. The definition of the models is given below.[8]

DEFINITION 4.1. **Action Model with Issues**

For $i > 0$, an AMLI action model of level i is a triple $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{pre}\rangle$, where:

- S is a finite domain of actions;
- For each $a \in \mathcal{A}$, Δ_a is a function that maps each action to a non-empty downward closed set of sets of actions;
- $\mathsf{pre} : \mathsf{S} \to \mathcal{L}_!^{\mathsf{AMLI}_i}$ is a function that assigns a precondition $\mathsf{pre}(\mathsf{x}) \in \mathcal{L}_!^{\mathsf{AMLI}_i}$ to each action $\mathsf{x} \in \mathsf{S}$.

Let $\delta_a(\mathsf{x}) := \bigcup \Delta_a(\mathsf{x})$, the set of actions indistinguishable from x to a.

We define a new update procedure, which is much more straightforward than the one for AMLQ, since we only need to take care of two things:

(i) Preserve the knowledge and issues from the original epistemic model (i.e. with respect to *worlds*). Formally, $\pi_1(s) \in \Sigma_a(w)$.

[8] Since the content of an action is always a statement in AMLI, content and precondition are always equivalent. Therefore, we do not need a separate function content here.

(ii) Preserve the knowledge and issues from the action model (i.e. with respect to *actions*). Formally, $\pi_2(s) \in \Delta_a(\mathsf{x})$.

DEFINITION 4.2. **Product update**
Let M be an inquisitive epistemic model and M an AMLI action model. Then $M' = (M \otimes \mathsf{M})$ is the product update of M and M, defined as follows.

$M' = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V' \rangle$, where:

- W' and V' are defined as usual.
- $s \in \Sigma'_a(\langle w, \mathsf{x} \rangle)$ iff
  (i) $\pi_1(s) \in \Sigma_a(w)$
  (ii) $\pi_2(s) \in \Delta_a(\mathsf{x})$

Since we require factivity of Σ_a, we need to require that the agent's knowledge about the actions is also factive (for all $\mathsf{x} \in \mathsf{S}$, $\mathsf{x} \in \delta_a(\mathsf{x})$), to make sure that factivity is preserved by the update. Similarly, whenever introspection is assumed in Σ_a, it is preserved by the update procedure as long as Δ_a satisfies introspection.

As for the semantics, we define the support condition for dynamic modalities analogous to AMLQ. The facts and definitions we discussed in Section 3.3 (persistence and empty state properties, declarative fragment, resolutions and normal form) carry over to AMLI.

4.2 Example

Let us look at a simple scenario that we can model in this system. Suppose there is a disease for which no cure is known yet. In order to find out whether two newly developed medicines cure this disease, an experiment takes place. However, the experiment is designed in such a way that for each medicine, the outcome can only be positive or indeterminate.

Thus, the experiment has four possible results: both medicines work ($p \wedge q$), one of them works (p or q) or the result is indeterminate for both medicines ($\top$). The action model contains four actions, one for each of these results.

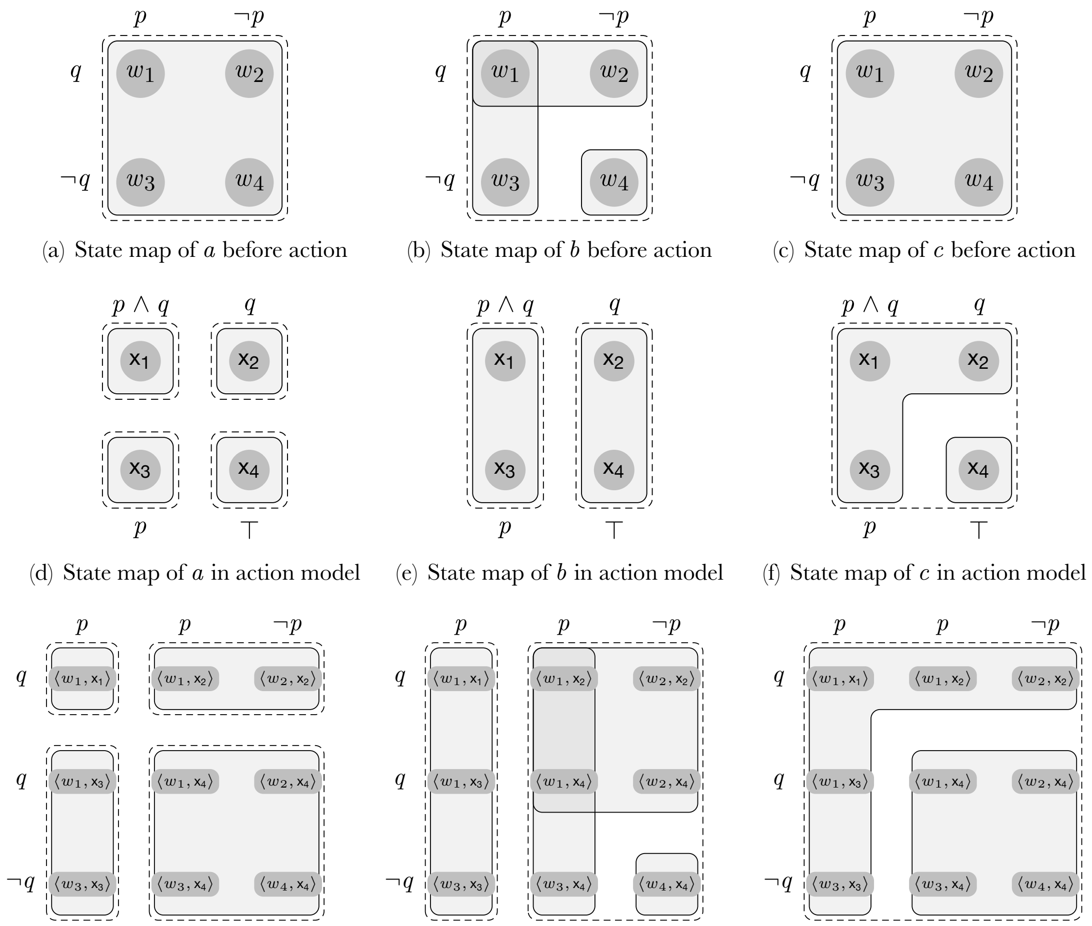
The experiment is performed by a laboratory technician (agent a), who is not involved in the development of the medicines and has no prior knowledge or non-trivial issues about whether the medicines work (Figure 9(a)). Being the observer herself, she knows exactly what is observed (Figure 9(d)).

There are also two other agents involved, who are not present when the experiment is performed, but are interested in the outcome. Agent b is a doctor, who wants to have a medicine to cure the disease. This means she already has an issue before the experiment, which is resolved either when one such medicine is found or when it becomes clear that neither of them works (Figure 9(b)).[9] Because information about q cannot be disclosed to medical professionals immediately after the experiment, the doctor only gets information about p, and not about q (Figure 9(e)).

Agent c is another laboratory technician and a colleague of a. In case the result of the experiment is indeterminate, she will be given the task to design a new experiment. Initially, she has no information about the outcome. Her issue is resolved by finding out whether or not observation x_4 was made (Figure 9(f)).

As can be seen in the updated model (Figure 9(g)-9(i)), the update results in state maps that combine the state maps from the original model and the action model. We can express the effects

[9] Notice that by this experiment it cannot be discovered that neither of the medicines work.



(a) State map of a before action (b) State map of b before action (c) State map of c before action

(d) State map of a in action model (e) State map of b in action model (f) State map of c in action model

(g) State map of a after action (h) State map of b after action (i) State map of c after action

Figure 9: Example of an update in AMLI

of the experiment on the knowledge and issues of the agents using dynamic modalities. Let s be the set of worlds in the original model and s the set of actions.

Agent a will not develop any issues. She will find a medicine just in case the observation she makes is not x_4. Thus, if we let $\mathsf{t} = \{\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3\}$, then we have $s \models [\mathsf{t}]K_a(p \mathbin{\vee\!\!\!\vee} q)$, while $s \models [\mathsf{x}_4]\neg K_a(p \mathbin{\vee\!\!\!\vee} q)$.

Because of the limited information agent b gets, her issue is resolved just in case the experiment shows that p is the case. This can be expressed as follows. Let $\mu = p \mathbin{\vee\!\!\!\vee} q \mathbin{\vee\!\!\!\vee} \neg(p \vee q)$. Then, before the experiment, agent b wonders what the answer to this question is: $s \models W_b\mu$. Let $\mathsf{t} = \{\mathsf{x}_1, \mathsf{x}_3\}$ and $\mathsf{t}' = \{\mathsf{x}_2, \mathsf{x}_4\}$. Then, $s \models [\mathsf{t}]\neg W_b\mu$ and $s \models [\mathsf{t}']\, W_b\mu$.

For agent c, the experiment generates an issue which is resolved by finding out whether the result was indeterminate or not. We can express this in terms of the knowledge of a. Let $\alpha = K_a(p \mathbin{\vee\!\!\!\vee} q)$. Then α is true just in case a knows of at least one medicine that it works. Thus, $s \models [\mathsf{s}]\, W_c?\alpha$.

4.3 Reduction and axiomatization

We have seen in Section 3.4 that every formula of AMLQ can be reduced to a formula of IEL. The same is the case for AMLI. All reduction equivalences we formulated for AMLQ carry over to AMLI, with one exception: the reduction equivalence for the entertain modality is different.

PROPOSITION 4.1. $[\mathsf{x}]E_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi$

Proof. We give a sketch of the proof. First, by unpacking Definition 4.1 it is easy to show that, instead of Lemma 3.2, we now have:

$$s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle) \iff s' \subseteq s[\mathsf{s}] \text{ for some } s \subseteq W \text{ and } \mathsf{s} \subseteq \mathsf{S} \text{ such that:}$$

- $s \in \Sigma_a(w)$
- $\mathsf{s} \in \Delta_a(\mathsf{x})$

With this fact, we can prove the equivalence by unpacking the formulas by the support conditions of the connectives, similar to the proof for Proposition 3.16. □

This characterizes the fact that the two systems are the same, except for the way in which new issues can be raised by an action. The proof system for AMLI is the same as the one for AMLQ, but with a different reduction axiom for the entertain modality, corresponding to the equivalence above. The completeness proof is analogous.

5 Comparison

In this section, we will compare the systems AMLQ and AMLI with the systems we discussed in the introduction and with each other.

5.1 Comparing AMLQ with AML and IDEL

Let us first see how AMLQ relates to AML. We have seen that the update procedure of AMLQ is standard with respect to knowledge. Furthermore, we can check that all connectives of the language of AML have the same truth conditions as those of AMLQ and that all $\alpha \in$ AML are truth-conditional. It follows that the semantics of these formulas remains standard in our setting, and so does the notion of entailment restricted to these formulas. This makes AMLQ a conservative extension of AML.

We have seen how IDEL extends IEL with public announcements. In AMLQ, the public announcement of a formula φ can be encoded in an action model with pub^φ as its only action and $\mathsf{content}(\mathsf{pub}^\varphi) = \varphi$. It can be shown that the effect of updating with this model in AMLQ is the same as the effect of the public announcement of φ in IDEL. Thus, if we define $[\varphi]\psi$ as an abbreviation of $[\mathsf{pub}^\varphi]\psi$, AMLQ is a conservative extension of IDEL. Like IDEL, it is then also a conservative extension of IEL and PAL. See Figure 10 for an overview of how these logics relate. Arrows to the left stand for an extension of the dynamic possibilities in the system; arrows to the right stand for making the logic inquisitive.

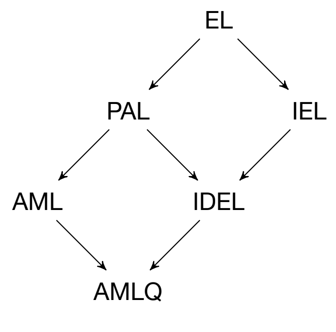


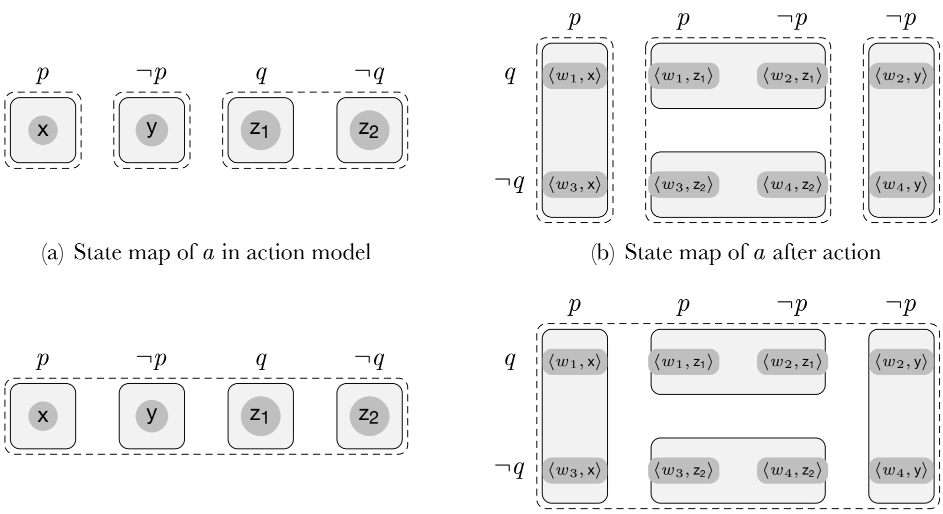
Figure 10: Relation between dynamic and inquisitive epistemic logics

5.2 Comparing AMLI with AMLQ

We will now compare the two systems developed in this paper. First, we observe that actions in AMLQ can be simulated in AMLI, while the converse is not the case. Then, we will see that combining the two systems poses a challenge.

5.2.1 Simulating AMLQ in AMLI

Let us consider once again the scenario from Section 3.2, in which our agents considered three speech acts to be possible: two of stating and one of asking. If we change our perspective slightly, we will see that this scenario can be simulated in AMLI. We have to allow actions to not just be observations, but also *future* observations. Then, the asking of a question $?q$ can be modelled by two actions which represent future observations that resolve $?q$: thus, one action for each resolution, q and $\neg q$, and an issue over them (Figure 11(a) and 11(c)). This is the same way that questions enter the system in ELQm ([22]).



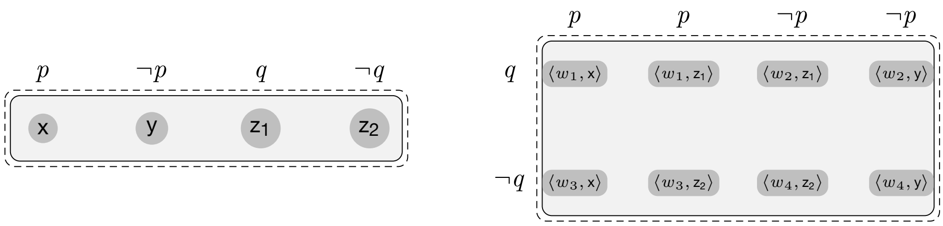
(a) State map of a in action model
(b) State map of a after action
(c) State map of b in action model
(d) State map of b after action

Figure 11: Example from Section 3.3 in AMLI

As can be seen, the resulting product update is isomorphic to the product update in the same example in AMLQ. Thus, it does not matter for the outcome whether we encode our scenario in AMLQ or AMLI.

Recall that in AMLQ, we made the assumption that uncertainty about epistemic actions is always an issue to the agents: if they don't know which action is the actual one, then their issues can only be resolved by finding this out. Now, let us add a third agent c, who is aware of the phone call, but not interested (Figure 12(a)). As we can see in the updated state map (Figure 12(b)), this agent will not develop any new issues. Thus, contrary to AMLQ, agents who are uninterested in the action can be encoded in an AMLI action model.

The example above is not just a special case: in fact, it turns out that any action model of AMLQ can be translated to an AMLI action model that gives an isomorphic product update. Let us see how.



(a) State map of c in action model

(b) State map of c after action

Figure 12: Uninterested agent in AMLI

DEFINITION 5.1. **Translation of AMLQ action model to AMLI action model**
Let $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{content}\rangle$ be an AMLQ action model. We will define a corresponding AMLI action model M^*.

First, we define a function C that maps each action to the set of all combinations of resolutions of its content. For every action $\mathsf{x} \in \mathsf{S}$, let $\varphi^{\mathsf{x}} \in \mathcal{L}^{\mathsf{IEL}}$ be a formula that is equivalent to $\mathsf{content}(\mathsf{x})$.[10]

$$\text{For every } \mathsf{x} \in \mathsf{S}, \text{ let } C(\mathsf{x}) := \wp(\mathcal{R}(\varphi^{\mathsf{x}})) - \emptyset$$

By definition, every member of $C(\mathsf{x})$ is a set of statements. For each set $C(\mathsf{x})$, let $C(\mathsf{x}) = \{\Gamma_1^{\mathsf{x}}, \dots, \Gamma_n^{\mathsf{x}}\}$. Then let $\overline{\Gamma_i^{\mathsf{x}}} = \mathcal{R}(\varphi^{\mathsf{x}}) - \Gamma_i^{\mathsf{x}}$ be the set of resolutions of the content of x that are not in Γ_i^{x} for this particular i.

Then we define the AMLI action model $\mathsf{M}^* = \langle \mathsf{S}^*, \{\Delta_a^* \mid a \in \mathcal{A}\}, \mathsf{pre}^*\rangle$ as follows:

- $\mathsf{S}^* := \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{S} \text{ and } \Gamma_i^{\mathsf{x}} \in C(\mathsf{x})\}$
- $\mathsf{s} \in \Delta_a^*(\mathsf{x}_i)$ iff
  - For all $\mathsf{y}_j \in \mathsf{s} : \mathsf{x} \sim_a \mathsf{y}$
  - For all $\mathsf{y}_j, \mathsf{y}'_k \in \mathsf{s} : \mathsf{y} = \mathsf{y}'$
  - There is some α such that for all $\mathsf{y}_j \in \mathsf{s} : \alpha \in \Gamma_j^{\mathsf{y}}$
- For every $\mathsf{x}_i \in \mathsf{S}^*$, $\mathsf{pre}^*(\mathsf{x}_i) := \bigwedge \Gamma_i^{\mathsf{x}} \wedge \bigwedge_{\alpha \in \overline{\Gamma_i^{\mathsf{x}}}} \neg\alpha$

It is easy to check that M^* is indeed an action model with issues according to Definition 4.1. We now show that updating any inquisitive epistemic model with either M or M^* gives us the same updated model.

PROPOSITION 5.1. **Isomorphism between updated models**
For every AMLQ action model M and its AMLI counterpart M^*, for every inquisitive epistemic model M : $M \otimes \mathsf{M}$ is isomorphic to $M \otimes \mathsf{M}^*$.

Proof. Take any AMLQ action model $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{content}\rangle$. Now take any inquisitive epistemic model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$. We claim that $M \otimes \mathsf{M} = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V'\rangle$ is isomorphic to $M \otimes \mathsf{M}^* = \langle W'', \{\Sigma''_a \mid a \in \mathcal{A}\}, V''\rangle$.

[10]Recall that $\mathsf{content}(\mathsf{x})$ may contain dynamic modalities of AMLQ action models. As we define an AMLI action model, they cannot be part of the content of our actions here. However, we can always find an equivalent formula $\varphi^{\mathsf{x}} \in \mathcal{L}^{\mathsf{IEL}}$ using Theorem 3.1.

First, observe that $\langle w, \mathsf{x}\rangle \in W'$ just in case $\langle w, \mathsf{x}_i\rangle \in W''$ for some i. To guarantee that we can make a bijection between any world $\langle w, \mathsf{x}\rangle \in W'$ and $\langle w, \mathsf{x}_i\rangle \in W''$, we also need to show that for any $\langle w, \mathsf{x}\rangle \in W'$, there is exactly one $\langle w, \mathsf{x}_i\rangle \in W''$. That is, for every $\langle w, \mathsf{x}_i\rangle, \langle w, \mathsf{x}_j\rangle \in W''$, it must be that $i = j$. To see that this is the case, suppose that $i \neq j$. Then by the definition of W'', $M, w \models \mathsf{pre}^*(\mathsf{x}_i)$ and $M, w \models \mathsf{pre}^*(\mathsf{x}_j)$. However, as $\Gamma^{\mathsf{x}}_i \neq \Gamma^{\mathsf{x}}_j$, there must be some α that is in one of these sets and not in the other. But then from the definition of pre^* we can see that w supports both this α and its negation. We have obtained a contradiction, so $i = j$.

From this we can conclude that if we let $f(\langle w, \mathsf{x}_i\rangle) = \langle w, \mathsf{x}\rangle$, then f is a bijection between W'' and W'. Now let us show that it is indeed an isomorphism. For this, we need to show two things:

(i) The mapping preserves the structure of the state maps. That is, if we write $\{f(w) \mid w \in s\}$ as $f(s)$, then we have:

$$s \in \Sigma''_a(\langle w, \mathsf{x}_i\rangle) \iff f(s) \in \Sigma'_a(f(\langle w, \mathsf{x}_i\rangle))$$

(ii) The mapping preserves the valuation: $V''(\langle w, \mathsf{x}_i\rangle) = V'(f(\langle w, \mathsf{x}_i\rangle))$

Given the definition of valuation, the latter is trivial, so we only show (i). Take any world $\langle w, \mathsf{x}_i\rangle \in W''$ and any state $s'' \subseteq W''$. Let $s' = f(s'')$. Then we need to show that $s'' \in \Sigma''_a(\langle w, \mathsf{x}_i\rangle) \iff s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

($\Rightarrow$) Assume $s'' \in \Sigma''_a(\langle w, \mathsf{x}_i\rangle)$.

Then s'' satisfies conditions (i) and (ii) of Definition 4.2 to be in $\Sigma''_a(\langle w, \mathsf{x}_i\rangle)$. We need to show that s' satisfies conditions (i)-(iv) of Definition 3.5 to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$. Since $\pi_1(s') = \pi_1(s'')$ and the first condition is the same, we only need to check conditions (ii), (iii) and (iv). If s' is empty we are done, so assume it is not. From the fact that s'' satisfies condition (ii) of Definition 4.2 we know that $\pi_2(s'')$ satisfies the three conditions to be in $\Delta^*_a(\mathsf{x})$ formulated in the definition of M^*. From the first condition of that definition we have that $\forall \mathsf{y} \in \pi_2(s') : \mathsf{x} \sim_a \mathsf{y}$, which means condition (ii) is satisfied.

We know that there is exactly one $\mathsf{y} \in \pi_2(s')$ because for all $\mathsf{y}_j, \mathsf{y}'_k \in \pi_2(s'') : \mathsf{y} = \mathsf{y}'$, so condition (iii) is satisfied. By definition there is some α such that for all $\mathsf{y}_j \in \pi_2(s'')$, $\alpha \in \Gamma^{\mathsf{y}}_j$. Now take any $\langle v, \mathsf{y}_j\rangle \in s''$. Then $M, v \models \mathsf{pre}^*(\mathsf{y}_j)$, so $M, v \models \alpha$. As $\langle v, \mathsf{y}_j\rangle$ was chosen arbitrarily, this goes for all $v \in \pi_1(s'')$, so $M, \pi_1(s'') \models \alpha$ by the truth-conditionality of α. As $\alpha \in \mathcal{R}(\varphi^{\mathsf{y}})$, we have $M, \pi_1(s'') \models \varphi^{\mathsf{y}}$ and therefore $M, \pi_1(s'') \models \mathsf{content}(\mathsf{y})$. So condition (iv) is satisfied too.

As s satisfies conditions (i)-(iv) of Definition 3.5, $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

($\Leftarrow$) Assume $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

Then s' satisfies conditions (i)-(iv) of Definition 3.5 to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$. We need to show that s'' satisfies conditions (i) and (ii) of Definition 4.2 as well. As the first condition is the same, we only need to check that $\pi_2(s'') \in \Delta^*_a(\mathsf{x})$.

In our definition of M^*, we formulated three conditions for a set of actions to be in $\Delta^*_a(\mathsf{x})$. These are all satisfied if s'' is empty, so suppose it is not. We have for all $\mathsf{y}_j \in \pi_2(s'') : \mathsf{x} \sim_a \mathsf{y}$ by condition (ii) of Definition 3.5. By condition (iii) there is just one $\mathsf{y} \in \pi_2(s')$. This guarantees us that for all $\mathsf{y}_j, \mathsf{y}'_k \in \pi_2(s'') : \mathsf{y} = \mathsf{y}'$.

That leaves only the third condition. By condition (iv), $M, \pi_1(s') \models \mathsf{content}(\mathsf{y})$. This means that $M, \pi_1(s'') \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi^{\mathsf{y}})$. Now take any world $\langle v, \mathsf{y}_j\rangle$ in s''. Suppose for reductio that $\alpha \notin \Gamma^{\mathsf{y}}_j$. By definition, $\alpha \in \overline{\Gamma^{\mathsf{y}}_j}$. Then from $M, v \models \mathsf{pre}^*(\mathsf{y}_j)$ we obtain

$M, v \models \neg\alpha$. But as v is in $\pi_1(s'')$ and $M, \pi_1(s'') \models \alpha$, we have $M, v \models \alpha$, so we have a contradiction. This means $\alpha \in \Gamma_j^{\mathsf{y}}$. Because the world was chosen arbitrarily, this goes for all worlds in s''. Hence, $\pi_2(s'')$ satisfies the third condition.

As $\pi_2(s'')$ satisfies all three conditions to be in $\Delta_a^*(\mathsf{x})$, s'' satisfies condition (ii) of Definition 4.2, which means $s'' \in \Sigma_a''(\langle w, \mathsf{x}_i \rangle)$.

We have thereby shown that f is indeed an isomorphism between $M \otimes \mathsf{M}$ and $M \otimes \mathsf{M}^*$. □

This means that every action in AMLQ can be simulated in AMLI. However, the converse is not the case, as we have already seen, because in AMLI we can encode agents who are uninterested about which action was performed, while these cannot be encoded in AMLQ.

5.2.2 Combining AMLI with AMLQ

As we have seen, the system AMLI is similar to AMLQ. The fact that there is a translation from AMLQ to AMLI means the latter inherits the conservativity results of the former with respect to AML and IDEL. What makes AMLI more general than AMLQ is that issues about actions are encoded in an explicit way, allowing for agents that do not develop new issues as to which action was performed. Another advantage is that the update procedure of AMLI is symmetric in its treatment of worlds and actions.

However, we have also seen that if we interpret actions as speech acts, the AMLQ approach is more natural, since asking a question is treated in the same way as uttering a statement. This is especially relevant considering our initial goal of generalizing IDEL, because one of its key features is the way it treats statements and questions on a par.

It is therefore natural to ask whether we can combine the two approaches to obtain a system with both advantages. At first glance, it seems this would simply amount to interpreting actions as speech acts again, and thus allowing questions to be the contents of actions in AMLI. Let us sketch why this is more complicated than it may seem. In such a combined system, a new class of scenarios can be encoded: namely, scenarios in which an agent does not know which of two questions was asked, and does not care either. In such scenarios, there is a conflict between the issue raised by the content and the issue raised by the structure of the action model. The questions demand an issue, while the structure of the action model does not allow for one.

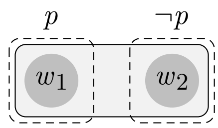
Although a combined system that is backwards compatible with both AMLQ and AMLI is surely possible, such conflicts make it difficult to have clear intuitions about what it should produce in cases that go beyond the scope of the two systems we have discussed in this paper. A principled solution to this problem, that does not require *ad hoc* conditions, is not yet available. It seems therefore that AMLQ and AMLI remain two separate systems. Both encode almost the same things, but in a different way. Depending on the interpretation we have in mind, we will either need questions on the level of actions (what was asked) or issues with respect to actions (what the agent wants to know). If we think about actions as speech acts, which is also the intended interpretation of IDEL, then AMLQ is the more natural choice.

5.3 Comparing AMLI with ELQm

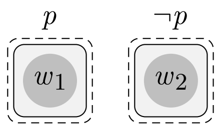
In the introduction we mentioned the Dynamic Logic of Questions (DELQ) and we noted some differences between this system and IDEL. We mentioned that there is a non-public variant of DELQ called ELQm, developed in [22]. This system is similar to AMLI, but has some important differences as well, which we will briefly review.

The static systems IEL and ELQ, which form the basis of AMLI and ELQm, respectively, have already been compared extensively in [8]. The most important differences, as we mentioned in the introduction, are the notion of issues, which is strictly more general in IEL, and the way questions enter the logical system. While in IEL there is a direct way of talking about the questions an agent entertains ($E_a\mu$), in ELQ a paraphrase is required. For instance, $U(Q_a\varphi \vee Q_a\neg\varphi)$ expresses that agent a entertains the question $?\varphi$ (see [22, p. 637]). This paraphrase uses the universal modality U. As a consequence of this approach, issues are always common knowledge, and it is impossible to express uncertainty about issues or issues about issues. These two differences between the static systems are inherited by their dynamic extensions AMLI and ELQm.

Another difference between IEL and ELQ is that in the latter framework, cells of issue partitions may extend beyond the boundaries of cells of epistemic partitions, as is the case in Figure 13(a). In this model, the issue relation encodes that the agent is indifferent as to whether p is the case. According to the epistemic indistinguishability relation, she *knows* whether p is the case. In this way, the model keeps track of what the agent's issues were prior to knowing whether p.



(a) Unrefined issue relation



(b) Refined issue relation

Figure 13: Unrefined and refined issue relations in ELQ

If we are interested in the agent's *current* issues, taking into account what she already knows, we only have to look at the issue cells within the boundaries of epistemic cells. The dynamic extension of ELQ [22] provides an *issue refinement* action, which transforms the issue relation in such a way that the agent's current issues become represented. The issue relation depicted in Figure 13(a) would be like Figure 13(b) after such a refinement action.

Although it is natural to keep track of prior and current issues in a setting in which information is treated as defeasible (for such a strategy in a doxastic version of IEL, see [20]), in AMLI and ELQm it is assumed that when we obtain new information, the contrary is ruled out irreversibly. Thus, it does not seem necessary to keep issues about this information around. Furthermore, it is not clear what the issue refinement action in ELQm corresponds to from a cognitive point of view.

In contrast, in IEL models, alternatives cannot include epistemically distinguishable worlds, which means that issues are always refined. Hence, obtaining new information can affect issues directly. Since action models in ELQm and AMLI have the same structure as their respective epistemic models, issues are encoded in action models in the same way. This means that, unlike in AMLI, ELQm action models can have unrefined issues. It is unclear to us how we should think of such issues.

In one respect, action models of AMLI are more general, because they allow non-partition issues. In another respect, ELQm action models are more general, since they allow for unrefined issues. Setting the differences between issues in ELQm and AMLI aside, we find that they share a common core, namely, refined partition models. Restricting ourselves to epistemic models and action models of this kind, we can compare the update procedures of ELQm and AMLI, by asking when it is the case that two worlds $\langle w, \mathsf{x}\rangle$ and $\langle w', \mathsf{x}'\rangle$ are in the same issue cell in the product update. This is the case if and only if in the original model, w and w' are in the same issue cell and in the action model, x and x' are.[11] Thus, in this common core, the updates in ELQm and AMLI coincide.

[11] The update procedures of ELQm and AMLI can only be compared in such a way because of the restriction to refined partition models. If AMLI-models are allowed to be non-partition models, they cannot be characterized by just looking at pairs of worlds. Analogously, if ELQm-models are allowed to be unrefined, then looking at just the issue relation is not enough.

6 Conclusion and further work

The objective of this paper was to generalize IDEL in such a way that private actions can be encoded, by merging its static basis IEL with AML. We have shown that this can be done in at least two ways: AMLQ and AMLI. The former is a very natural extension of IDEL in which asking a question is an epistemic action, just like uttering a statement. This is not the case in AMLI, but this system has different advantages, namely the fact that it can explicitly encode issues with respect to actions and its symmetric treatment of worlds and actions. Which of the two is more natural depends in part on the intended interpretation of epistemic actions. Both are conservative extensions of AML and IEL. We gave a sound and complete axiomatization for both, based on reduction axioms that allow us to turn each formula into a formula of the static language of IEL.

We have illustrated a challenge to the integration of these systems. We leave it to further work to find other approaches to action models with issues over actions *and* questions as content. In addition, there are several other directions for further research we consider.

So far, we have restricted ourselves to cases in which agents always consider the actual world (and action) possible. Doxastic cases, in which agents can be mislead, are often more exciting. This requires an integration with doxastic logic and belief revision. A first step in this direction is [20].

The logics developed in this paper only describe single actions of stating and asking. Much more can be said when looking at *sequences* of utterances in a temporal setting. Insights from this paper may enhance existing frameworks that investigate question-answer protocols (cf. [21], [22] for a system based on ELQ and [11] for an inquisitive approach). In such a setting, we would want to incorporate notions of common knowledge and *common issues* ([5]). Common issues can be taken to explain why certain utterances are appropriate in some context.

Another interesting line of research is that of question-answer games, either among agents ([1]), between an inquirer and nature (the Interrogative Model of Inquiry, [14, 15]), or among nature and several agents ($\mathsf{DEL}_{\mathsf{IMI}}$, [12]). We may be able to model similar situations in a setting based on AMLQ/AMLI, benefiting from the inquisitive view on issues and questions.

We may also investigate *pragmatics* in the AMLQ/AMLI setting, by taking *appropriateness conditions* for making statements and asking questions into account (cf. [5, Section 8.5]). We can extend the preconditions of a question μ to incorporate pragmatic assumptions about the speaker a, such as the fact that the speaker wonders about the question ($W_a\mu$).

With this paper, we hope to have provided a suitable logical foundation for these enterprises.

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A Appendix

A.1 Proof of Theorem 3.1

As mentioned before, the lack of a reduction equivalence for formulas of the form $[\mathsf{s}](\varphi \to \psi)$ can be overcome by the fact that $[\mathsf{s}](\varphi \to \psi)$ is equivalent to its normal form, $\bigvee \mathcal{R}([\mathsf{s}](\varphi \to \psi))$. Since the resolutions of $[\mathsf{s}](\varphi \to \psi)$ are guaranteed to be declaratives, they can be reduced using Proposition 3.7 and Proposition 3.13.

However, as can be seen from the definition of resolutions, the resolutions of a question of the form $\varphi \to \psi$ are more complex than $\varphi \to \psi$ itself. This means a straightforward induction on the structure of formulas is not possible. To be able to perform a straightforward induction, we require a complexity measure c that has the following properties:

1. If ψ is a proper subformula of φ, then $\mathsf{c}(\varphi) > \mathsf{c}(\psi)$
2. If φ is a not a declarative, then $\alpha \in \mathcal{R}(\varphi)$ implies $\mathsf{c}(\varphi) > \mathsf{c}(\alpha)$

The standard measure of complexity has the first property, but not the second. The following non-standard complexity measure does have both properties:[12]

- $\mathsf{c}(p) = \mathsf{c}(\bot) = 1$
- $\mathsf{c}(\varphi \circ \psi) = 1 + \mathsf{max}(\mathsf{c}(\varphi), \mathsf{c}(\psi))$ for $\circ \in \{\wedge, \bigvee\}$
- $\mathsf{c}(\blacksquare\varphi) = 1 + \mathsf{c}(\varphi)$ for $\blacksquare \in \{K_a, E_a, [\mathsf{s}]\}$
- $\mathsf{c}(\varphi \to \psi) = \begin{cases} 1 + \mathsf{max}(\mathsf{c}(\varphi), \mathsf{c}(\psi)) & \text{if } \varphi \to \psi \text{ is a declarative;} \\ 1 + |\mathcal{R}(\varphi)| + \mathsf{max}(\mathsf{c}(\varphi), \mathsf{c}(\psi)) & \text{otherwise.} \end{cases}$

By inspecting the definition, it is immediate that c has the first property. The second property can be shown by induction on the standard complexity of formulas.

Proof. We give only the crucial step, namely the one for implication. Take an arbitrary resolution of $\varphi \to \psi$. By the definition of resolutions, this will be a formula of the form $\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to \beta_\alpha)$ where, for each α, $\beta_\alpha \in \mathcal{R}(\psi)$. By the definition of c, its complexity is, at most, the amount of conjuncts minus 1, plus the complexity of some declarative implication $\alpha \to \beta$ where $\alpha \in \mathcal{R}(\varphi)$ and $\beta \in \mathcal{R}(\psi)$. Thus, at most, $|\mathcal{R}(\varphi)| - 1 + 1 + \mathsf{max}(\mathsf{c}(\alpha), \mathsf{c}(\beta)) = |\mathcal{R}(\varphi)| + \mathsf{max}(\mathsf{c}(\alpha), \mathsf{c}(\beta))$.

By the induction hypothesis it holds for both φ and ψ that, if they are not declaratives, their resolutions are strictly less complex. We also know that if a formula is a declarative, it has itself as its only resolution. Therefore, $\mathsf{c}(\alpha) \leq \mathsf{c}(\varphi)$ and $\mathsf{c}(\beta) \leq \mathsf{c}(\psi)$. Hence, the complexity of our arbitrary resolution cannot be bigger than $|\mathcal{R}(\varphi)| + \mathsf{max}(\mathsf{c}(\varphi), \mathsf{c}(\psi))$. By definition of c, $\mathsf{c}(\varphi \to \psi) = 1 + |\mathcal{R}(\varphi)| + \mathsf{max}(\mathsf{c}(\varphi), \mathsf{c}(\psi))$. Thus $\varphi \to \psi$ has property 2. □

We proceed by showing that all formulas of $\mathcal{L}^{\mathsf{AMLQ}_1}$ can be reduced. Take any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_1}$. We need to show that there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$. We perform an induction on the structure of φ. All steps are immediate, except for the one where φ is $[\mathsf{s}]\psi$. By the induction hypothesis we have a $\psi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\psi \equiv \psi^*$, so $\varphi \equiv [\mathsf{s}]\psi^*$. Therefore, what we need to show is that $[\mathsf{s}]\psi^* \equiv \varphi^*$ for some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$. This we do by induction on $\mathsf{c}(\psi^*)$.

[12] Here $|\mathcal{R}(\varphi)|$ is the number of resolutions of φ. By definition, this is always a finite number.

- **Base case.** ψ^* is an atom p or $\bot$. Take any set of actions t. Then by Proposition 3.7 and Proposition 3.8 or 3.9 we have:

  $[\mathsf{t}]\psi^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{t}}(\mathsf{pre}(\mathsf{x}) \to \psi^*)$

  Which means that in case of $[\mathsf{s}]\psi^*$ we can let $\varphi^* := \bigwedge_{\mathsf{x}\in\mathsf{s}} \mathsf{pre}(\mathsf{x}) \to \psi^*$.

- **Inductive step.** Induction hypothesis: for all χ less complex than ψ^*, for all sets of actions t, there is some $\chi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $[\mathsf{t}]\chi \equiv \chi^*$.

  ($\wedge$) Suppose ψ^* is $\chi \wedge \chi'$. Then by Proposition 3.10, $[\mathsf{s}]\psi^* \equiv [\mathsf{s}]\chi \wedge [\mathsf{s}]\chi'$. By the induction hypothesis, we have some $\chi^*, \chi'^* \in \mathcal{L}^{\mathsf{IEL}}$ which are equivalent to $[\mathsf{s}]\chi$ and $[\mathsf{s}]\chi'$ respectively. So we can let $\varphi^* := \chi^* \wedge \chi'^*$.

  ($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$) Analogous to conjunction.

  ($\to$) Suppose ψ^* is $\chi \to \chi'$.

  Then $\chi \to \chi'$ is either a declarative or it is not. Let us first consider the former. Then by Proposition 3.7 and 3.13, $[\mathsf{s}]\psi^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}([\mathsf{x}]\chi \to [\mathsf{x}]\chi')$. Take any $\mathsf{x} \in \mathsf{s}$. By the induction hypothesis, we have some $\chi^*, \chi'^* \in \mathcal{L}^{\mathsf{IEL}}$ which are equivalent to $[\mathsf{x}]\chi$ and $[\mathsf{x}]\chi'$ respectively. Let $\theta_{\mathsf{x}} := \chi^* \to \chi'^*$. Then $\theta_{\mathsf{x}} \equiv [\mathsf{x}]\chi \to [\mathsf{x}]\chi'$. As we can define such a θ_{x} for all $\mathsf{x} \in \mathsf{s}$, we can let $\varphi^* := \bigwedge_{\mathsf{x}\in\mathsf{s}} \theta_{\mathsf{x}}$.

  Now suppose $\chi \to \chi'$ is not a declarative. In that case, the strategy above is not available, since there is no reduction equivalence for $[\mathsf{s}](\chi \to \chi')$. However, we can use the fact that all formulas are equivalent to their normal form.

  Take any $\alpha \in \mathcal{R}([\mathsf{s}]\psi^*)$. By definition, this α will be of the form $[\mathsf{s}]\beta$. Since β is a resolution of ψ^*, by the induction hypothesis we have some $\alpha^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\alpha^* \equiv [\mathsf{s}]\beta$. Thus, we let $\varphi^* := \mathbb{V}\{\alpha^* \mid \alpha \in \mathcal{R}([\mathsf{s}]\psi^*)\}$.

  (K) Suppose ψ^* is $K_a\chi$. Then by Proposition 3.7 and 3.15 we have $[\mathsf{s}]\psi^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\chi)$. By the induction hypothesis we have some $\chi^*_{\mathsf{x}} \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to each $[\delta_a(\mathsf{x})]\chi$. So we can let $\varphi^* := \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to K_a\chi^*_{\mathsf{x}})$.

  (E) Suppose ψ^* is $E_a\chi$. Then by Proposition 3.7 and 3.16 we have $[\mathsf{s}]\psi^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \to [\mathsf{y}]\chi))$. By the induction hypothesis we have some $\chi^*_{\mathsf{y}} \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to each $[\mathsf{y}]\chi$. So we can let φ^* be defined as:

  $$\bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{content}(\mathsf{y}) \to \chi^*_{\mathsf{y}}))$$

This concludes the proof that we can find a $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to $[\mathsf{s}]\psi^*$. As this was the only step which was left to prove, we have thereby shown that for every $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_1}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$.

We can now generalize this claim to all formulas of $\mathcal{L}^{\mathsf{AMLQ}}$, by an induction on the natural numbers.

- **Base case.** By definition $\mathcal{L}^{\mathsf{AMLQ}_0}$ is just $\mathcal{L}^{\mathsf{IEL}}$, and we have already shown that the claim holds for $\mathcal{L}^{\mathsf{AMLQ}_1}$.

- **Inductive step.** For $\mathcal{L}^{\mathsf{AMLQ}_i}$, the proof for the base case can be repeated. However, this time when we want to use $\mathsf{pre}(\mathsf{x})$ or $\mathsf{content}(\mathsf{x})$ of some action x in our reduction, it might be the case that these are not formulas of IEL. But then they are formulas of at most $\mathcal{L}^{\mathsf{AMLQ}_{(i-1)}}$, which means that by the induction hypothesis we can obtain an equivalent formula from IEL that we can use instead.

A.2 Proof of Theorem 3.2

We start by showing that every formula of $\mathcal{L}^{\mathsf{AMLQ}}$ is inter-derivable with an equivalent formula in $\mathcal{L}^{\mathsf{IEL}}$. As our inference rules for dynamic modalities correspond to the reduction equivalences shown in Section 3.4, we can follow the same strategy as in the proof for Theorem 3.1. However, this proof relies on the equivalence between formulas and their normal form, so we first need to show that every formula is also inter-derivable with its normal form.

LEMMA A.1. **Provability of normal form**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, $\varphi \dashv\vdash \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.

Proof. By induction on the complexity of φ. We can repeat the base case and the inductive step for $\bigvee\!\!\!\!\bigvee$, $\wedge$ and $\rightarrow$ from [5, p. 86]. The steps for the modalities K_a and E_a are trivial: by definition, for all modal formulas α, $\mathcal{R}(\alpha) = \alpha$. We only need to add the inductive step for dynamic modalities.

Suppose φ is $[\mathsf{s}]\psi$. By the induction hypothesis, we have $\psi \dashv\vdash \bigvee\!\!\!\!\bigvee \mathcal{R}(\psi)$. We can use RE to obtain:

$$[\mathsf{s}]\psi \dashv\vdash [\mathsf{s}] \bigvee\!\!\!\!\bigvee \mathcal{R}(\psi)$$

Also, using $!\bigvee\!\!\!\!\bigvee$ in both directions we can get:

$$[\mathsf{s}] \bigvee\!\!\!\!\bigvee \mathcal{R}(\psi) \dashv\vdash \bigvee\!\!\!\!\bigvee_{\alpha \in \mathcal{R}(\psi)} [\mathsf{s}]\alpha$$

By the definition of resolutions, we have the following equivalence:

$$\bigvee\!\!\!\!\bigvee_{\alpha \in \mathcal{R}(\psi)} [\mathsf{s}]\alpha \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}([\mathsf{s}]\psi)$$

This means we can combine the two inter-derivabilities to:

$$[\mathsf{s}]\psi \dashv\vdash \bigvee\!\!\!\!\bigvee \mathcal{R}([\mathsf{s}]\psi)$$

This concludes the inductive step for the dynamic modality, which was all we needed to show that $\varphi \dashv\vdash \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$. □

Now that we have provability of normal form, we can move on to the proof for the following theorem.

THEOREM A.1. **Every formula of AMLQ is inter-derivable with a formula of IEL**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \dashv\vdash \varphi^*$.

Proof. The proof for this claim is analogous to the proof for Theorem 3.1, but using the relation of inter-derivability instead of logical equivalence. Instead of the reduction equivalences, we use the corresponding deduction rules from Figure 8 and RE for substitution of equivalents under dynamic modalities. We use Lemma A.1 instead of Proposition 3.5. □

Now, notice that we have conservativity of entailment in AMLQ over IEL:

PROPOSITION A.1. **Entailment in AMLQ is conservative over IEL**
For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IEL}}$, $\Phi \models_{\mathsf{AMLQ}} \psi \iff \Phi \models_{\mathsf{IEL}} \psi$.

Proof. This is immediate from the fact that AMLQ has standard support conditions for all the connectives that are in the syntax of $\mathcal{L}^{\mathsf{IEL}}$. □

This means that we can use the entailment relation $\models$ for entailment in IEL and AMLQ interchangeably. Now we can finally prove the soundness and completeness of $\vdash$.

($\Rightarrow$) Suppose $\Phi \models \psi$. Then by Theorem 3.1, for every $\varphi \in \Phi$ there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi^* \equiv \varphi$ and some $\psi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\psi^* \equiv \psi$. Let Φ^* be a set containing for each $\varphi \in \Phi$ a formula φ^* equivalent to φ. Then we have $\Phi^* \models \psi^*$. We obtain $\Phi^* \vdash \psi^*$ by the completeness of the proof system for IEL. The previous theorem guarantees that $\Phi \vdash \varphi^*$ for all $\varphi^* \in \Phi^*$ and $\psi^* \vdash \psi$. This means that $\Phi \vdash \psi$.

($\Leftarrow$) The inference rules consist of the inference rules in Figure 8 and the ones of IEL. The former are sound by Proposition 3.7 and 3.8 - 3.16 and the latter by Proposition 7.3.11 of [5, p. 283].